Convergence of Stationery Solutions of Reaction- Diffusion Problems

Panayiotis M. VLAMOS
Dept. of Mathematics, National Technical Univ. of Athens, Zografou Campus 15780

Abstract: *We study the convergence of the sequence of the classical stationary solutions of the typical reaction-diffusion model, under specific restriction in the domain and the value of the positive parameter involved.*
Using the integral transformation of the problem, we obtain the convergence of this sequence towards a weak limitting regular solution.
Keywords*: reaction-diffusion, convergence, stationary solution, limitting regular solution.*

1 Introduction & Physical Model

We shall consider the steady-state problem of the classical model for reaction-diffusion processes:

$$\begin{Bmatrix} Lw + l_\infty f(w) = 0, x \in \Omega \\ Bw = 0, x \in \partial\Omega \end{Bmatrix} \qquad (1.1)$$

where L is the uniform elliptic operator, Ω a bounded domain of IR^N, $3 \le N \ge 9$, with a smooth boundary $\partial\Omega$ and B the operator which expresses the boundary conditions of the problem.

Let f satisfies the following properties :

i) $f(u) > 0,\ f'(u) > 0,\ f''(u) > 0,$ for every u

ii) $\int_b^\infty \frac{ds}{f(s)} < \infty$

Then, we already know (Amann [1,2,3,4]) that there exists a value λ_∞ of the positive parameter λ, for witch the problem (1.1) has infinite classical solutions $\{w_k(x)\}$.

We are concerned with the convergence of this sequence of classical solutions towards a singular solution $w_\infty(x)$ of the problem (1.1).

We already know (Bebernes- Eberly [5], Chapter 2, Theorem 2.19), that for the Gelfand problem (Gelfand [7]), in the symmetric case :

$$u'' + \frac{N-1}{r}u' + de^u + 0, 0 < r < 1 \qquad (1.2a)$$

$$u'(0) = 0, u(1) = 0 \qquad (1.2\beta)$$

there exists a value $\dot{d} = 2(N-2)$ of the positive parameter d , for which there exist infinite classical solutions, with the last one being concave in the interval [0,1], with $u''(1) = 0$, while the rest being bell-shaped. We expect a similar behavior for the case of the general problem that we are considering.

2. Existance and Integrability of solutions

Let us consider the limit of the sequence of the classical solutions of (1.1) for $l = l_\infty$:

$$w_\infty(x) = \lim_{n\to\infty} w_k(x) a.e. \qquad (2.1)$$

We define the function:

$$g_k(x) = \max\{|w_1(x)|, |w_2(x)|, \ldots |w_k(x)|\} \qquad (2.2)$$

Then, we can consider the function:

$$g(x) = \lim_{n\to\infty} g_k(x) \qquad (2.3)$$

At first we shall prove the existence of this limit.

The sequence $\{g_k\}$ is a sequence of measurable functions – as they are all bounded – and moreover increasing, since:

$$0 \le g_1(x) \le g_2(x) \le \ldots \le g_k(x) \le \ldots, \text{ for } x \in \Omega \qquad (2.4)$$

Hence, the Monotone Convergence Theorem of Lebesgue implies that there exists the limit $g(x) = \lim_{n\to\infty} g_k(x)$ and additionally that :

$$\lim_{k\to\infty} \int_\Omega g_k = \int_\Omega (\lim_{k\to\infty} g_k) = \int_\Omega g$$

We then prove the following lemma.

Lemma

The function g(x) defined from the relation (2.3) is Lebesgue integrable.

Proof
We claim that g is bounded almost everywhere. Indeed, if we suppose that g is not bonded almost everywhere, then there should exist a subset $\Omega_1 \subseteq \Omega$ with $m(\Omega_1) \neq 0$, such that: $g/_{\Omega_1} \equiv \infty$

Since $\begin{array}{l} g(x) = \lim_{k\to\infty} g_k(x) = \\ \lim_{k\to\infty}\{\max\{|w_1(x)|,|w_2(x)|,...|w_k(x)|\}\} \end{array}$

And tends to infinity for $x \in \Omega_1$, we obtain that:

$$\forall m > 0, \begin{array}{l} \exists x_0 \in \Omega_1 : \forall x \in V(x_0) \subset \Omega_1 \Rightarrow \\ \max\{|w_1(x)|,|w_2(x)|,..|w_k(x)|\} \geq m \end{array} \quad (2.6)$$

This can not finally hold – in a subregion of V(x_0)- but only for the solution with the greater supremum, which is $w_k(x)$.
Consequently, we get
$|w_k(x)| \geq m$, for $x \in V_1(x_0) \subset V(x_0)$ with
$m(V_1(x_0)) \neq 0$ (2.7)

We the consider the first eigenvalue m_1 of the corresponding linearized problem :

$$\left\{\begin{array}{l} \Delta y + mf'(w_k(x))y = 0, x \in \Omega \\ By = 0, x \in \partial\Omega \end{array}\right\} \quad (2.8)$$

From the variational characterization of the first eigenvalue $m_1(l)$ (Crandall & Rabinowitz [6]), we get that

$$m_1(l_\infty) = \inf_{y\in M}\left[\frac{\langle y, -\Delta y\rangle}{\langle y, f'(w_k(x))y\rangle}\right] \quad (2.9)$$

where $\langle y, j\rangle = \int_\Omega y(x)j(x)dx$ and

$$M = \{y(x) / \begin{array}{l} y(x) > 0 in\Omega, y(x) \in \\ C(\overline{\Omega}) \cap C^1(\Omega), y(x) \equiv 0 in\partial\Omega_1 \end{array}\}$$

whereas in $\partial\Omega_1$, we have that $b(x) \equiv 0$ and $a(x \equiv 1)$, in the standard boundary operator B.

But for $l_\infty \in (0, l^*)$ we know (Keller & Cohen [8]) that $m_1(l_\infty)$ is decreasing for f convex, hence $m_1(l_\infty) > l^*$. If $w_k \to \infty$ for every $x \in V_1(x_0)$ with $m(V_1(x_0)) \neq 0$, since f' is increasing, we get: $f'(w_k(x)) \to \infty$ for every $x \in V_1(x_0)$ with $m(V_1(x_0)) \neq 0$.
But for $m_1(l_\infty) = 0$, which is a contradiction to the fact that $m_1(l_\infty) > l^* > 0$. We thus proved that g is Lebesgue integrable.

3. Convergence of stationary solutions
We now can prove the convergence of the stationary solutions with the following theorem.

Theorem
The function $w_\infty(x)$ is a weak limitting regular solution of the problem (1.1)

Proof
From the previous lemma we have that g is Lebesgue integrable and obviously
$|w_k(x)| \leq g(x)(k = 1,2,...)$, for every x.
Since f is increasing we obtain that

$$|f(w_k(x))| \leq |w_k(x)| \leq f(g(x))$$

Multiplying with the corresponding Green function we have that

$$|G(x,y)\cdot(f(w_k(y)))| \leq G(x,y)f(g(x))$$

We shall aply the Dominated Convergence Theorem of Lebesgue.
Let $f(x) =: \lim_{n\to\infty} f_n(x) a.e.$ and a Lebesgue integrable function g, such that
$|f_n(x)| \leq g(x)(n = 1,2,...)$, for every x.
Then f is also a Lebesgue integrable function and

$$\lim_{n\to\infty}\int|f_n - f| = 0 \Leftrightarrow \lim_{n\to\infty}\int f_n = \int f$$

Thus, we get

$$\lim_{k\to\infty}\int_\Omega G(x,y)f(w_k(x))dy = \int_\Omega G(x,y)[\lim_{k\to\infty} f(w_k(y))]dy = \int_\Omega G(x,y)f(w_\infty(y))dy \Leftrightarrow$$

$$\Leftrightarrow \lim_{k\to\infty} l_\infty \int_\Omega G(x,y)f(w_k(x))dy = l_\infty \int_\Omega G(x,y)f(w_\infty(y))dy \Leftrightarrow$$

$$\Leftrightarrow \lim_{k\to\infty} w_k(x) = l_\infty \int_\Omega G(x,y)f(w_\infty(y))dy \Leftrightarrow w_\infty(x) = l_\infty \int_\Omega G(x,y)f(w_\infty(y))dy$$

Hence w_∞ is a weak solution of the steady-state problem, its supremum is equal to ∞, so it is a limitting regular solution of the problem.

4. Discussion

A lot of aspects of the steady- state problem of the typical reaction-diffusion model are not yet entirely considered. This leads to questions which arise from the practical interest concerning the model. In cases where infinite solutions correspond to a positive value of the involved parameter, it is crucial to know their convergence.

Until now the fact that their limit was not a classical solution prevented the study of their convergence. Transferring the problem to an integral equation the convergence seems natural and the characterization of the limitting function as limitting regular solution describes exactly the situation of the model.

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