

The Computation of Bond Values by Simulating Transition Matrixes and Calibration of BDT

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Credit risk represents the risk of possible losses caused by unexpected changes in the credit ratings. This paper uses the methodology developed in Barnhill&Maxwell (2002) by computing a ten-year bond using 100,000 paths of Monte Carlo simulations for the transition matrixes. The main contribution consists in the development of the Matlab program that produces the transition matrixes and the calibration of the Black Derman Toy model for a ten-year risk-free term structure by the use of some imaginary term structure and yield volatilities.

Computing the transition probabilities

The first part of our program deals with the simulation of the market index that will be used for the computation of the risk premiums for each rating class through the CAPM. The model used for the simulation of the index is the geometric Brownian motion model

$$\frac{S + \Delta S}{S} = \exp \left[\left(m - \frac{\sigma^2}{2} \right) \Delta t + \sigma \varepsilon \sqrt{\Delta t} \right] \tag{1}$$

where ε is the random drawing from the standard normal distribution and the expected growth rate m and volatility σ are constant. The expected return on the equity index is the risk free rate plus a premium of 8% and the equity return volatility for the S&P 500 is set to 23% as specified in the Barnhill&Maxwell (2002). We used a risk free rate of 5%.

The value of the return on equity for all the rating classes is realized through the CAPM formula:

$$K_i = R_F + Beta_i (K_m - R_F) + \sigma_i \Delta z$$

where Δz is a Wiener process and the values for $Beta_i$ and σ_i are those provided in the paper for the high volatility firms. The market index is computed according to (1) to which we added the dividend yield of 2.6%. The result will be represented by 100 000

values for the possible returns on equity for each rating class – the same values for the random drawings in the Wiener process are used in both the market index return and the equity return.

Mean beta 1993–1998	Mean firm specific equity return volatility 1993–1998
0.682	0.317
0.757	0.363
0.864	0.412
0.994	0.507
1.131	0.729
1.314	0.727
1.301	0.954

Source: Barnhill & Maxwell (2003) pg. 359
The next part deals with the computation of the ratings. We assume that the company is changing the rating class when the debt ratio exceeds a certain level. The debt ratios are computed as the book value of debt divided by the sum of the book value of debt and the market value of equity. Barnhill & Maxwell (2003) provide a distribution of the debt ratios for all the rating classes over the period 1997 to 1998. Our purpose is to compute the new debt ratios by using the returns on equity simulated previously.

Debt ratios data

Rating	N	Mean	Std. dev.	Max.	Q ₃	Median	Q ₁	Min.
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Aaa	286	0.144	0.145	0.748	0.157	0.101	0.048	0.015
Aa	1067	0.163	0.120	0.690	0.204	0.127	0.077	0.017
A	3646	0.240	0.140	0.821	0.340	0.212	0.131	0.012
Baa	4312	0.319	0.159	0.832	0.431	0.305	0.198	0.011
Ba	3500	0.397	0.207	0.960	0.554	0.386	0.226	0.012
B	3076	0.515	0.235	0.983	0.702	0.525	0.324	0.015
Caa	34	0.729	0.262	0.984	0.931	0.819	0.615	0.117
Def	17	0.779	0.226	0.990	0.940	0.851	0.699	0.127

Source: Barnhill & Maxwell (2003) pg. 362

The first step is to compute the initial equity value from the initial debt ratio as presented in the table. If the initial debt ratio is d then we have

$$d = \frac{debt}{debt + equity}$$

Assuming that we start from a debt of 1 then

$initial\ equity = \frac{1-d}{d}$ and we can compute the new equity (one year from now) as $new\ equity = initial\ equity(1 + simulated\ ROE)$

. The new debt ratio will be computed as

$$d_{new} = \frac{1}{1 + new\ equity}$$

assuming that the book value of debt is kept constant for this period. The distribution of the debt ratios presented in the paper provides means, medians and quartiles. The authors suggest the use of the midpoint between the two quartiles as a proxy for the initial value for the debt ratio. We did the simulation of the debt ratios both by using the means as a starting value for the debt ratios but also by using the midpoint between the two quartiles. We obtain results closer to the Moody's and to those presented in the paper when the midpoints were used. The quartiles are overlapping and the midpoint is a choice that provides more realistic results but we can not say that the simulation methodology

presented before can generate the distribution of debt ratios.

The transition matrix was computed by counting all the situations in which the simulated debt ratio exceeded the quartile intervals. Because of the overlapping, the intervals for the other rating classes are smaller. For instance, when we compute the new debt ratios for the Baa rating class we start from the midpoint 0.3145 between the lower quartile 0.198 and the higher quartile 0.431. If the debt ratio exceeded 0.431 but not higher than 0.554 (which is the higher quartile for the debt ratio corresponding to the Ba rating class) then we record a downgrade to Ba rating class. We can see that all the other intervals are smaller than the interval between the two quartiles for the Baa rating class and the starting point is at the middle of this interval.

We then compose the transition matrix in which each cell contains the number of changes to the respective rating class divided by 100 000, the number of all the simulations.

We notice that the values for the transition probabilities in the Caa class are quite high for the rating classes in the closed neighborhood. The quartiles provided in the paper for the Caa class are for the whole C class companies.

Transition matrix where the starting point is the midpoint between the two quartiles

	Aaa	Aa	A	Baa	Ba	B	Caa	Default
Aaa	0.85625	0.07019	0.05093	0.00837	0.00531	0.00274	0.00078	0.00543
Aa	0.08212	0.7356	0.12426	0.02176	0.0136	0.00756	0.00217	0.01293
A	0.00659	0.08495	0.71923	0.07822	0.04664	0.02387	0.00736	0.03314
Baa	0.01122	0.01178	0.16852	0.581	0.091	0.05039	0.01577	0.07032
Ba	0.03111	0.00338	0.08485	0.07369	0.57418	0.0753	0.02462	0.13287
B	0.01189	0	0.00482	0.01272	0.1563	0.57935	0.04546	0.18946
Caa	5.00E-05	0	0	0	2.00E-05	0.22981	0.33095	0.43917

Transition matrix where the starting point is the mean of debt ratios

	Aaa	Aa	A	Baa	Ba	B	Caa	Default
Aaa	0.66631	0.16636	0.12568	0.01785	0.00979	0.00521	0.00161	0.00719
Aa	0.02717	0.72139	0.17552	0.03104	0.01744	0.00967	0.003	0.01477
A	0.00637	0.07444	0.72357	0.08124	0.04776	0.02584	0.00742	0.03336
Baa	0.01115	0.0096	0.15894	0.58866	0.09345	0.05155	0.01559	0.07106
Ba	0.02981	0.00237	0.07566	0.06891	0.58526	0.0779	0.02538	0.13471
B	0.01188	0	0.00429	0.01216	0.15425	0.58308	0.04505	0.18929
Caa	4.20E-04	0	0	0	1.02E-03	0.34283	0.27983	0.3759

A Matlab program used for the computation of this matrix could be used as a function to imply the parameters of the models used by using a Moody's matrix as an input. We could probably see the differences between the actual statistically computed parameters and those implied by the transition matrix assuming the models are right.

Calibrating the Black Derman Toy model

The next part of the project deals with the calibration of the BDT model according to the term structure provided in Barnhill & Maxwell (2003) and some imaginary yield volatilities for the Treasury short rates.

First we computed the yields for the 10 years implied by the 1, 5 and 10 year yields

provided in the paper for the risk free rates by using a nonlinear interpolation. Next we built a Matlab function for the 10-step binomial tree for the short rates using some starting values for the short rate volatilities. We used this function to estimate the values for the minimum rates and for the short rate volatilities that are appropriate for the term structure and the yield volatilities to which we try to do the calibration. Thus we estimated the values for these variables that minimize the differences between the real prices and the model prices. The results are presented in the following tables.

Short rates implied by the BDT model and the term structure and yield volatilities

1 st step	2 nd step	3 rd step	4 th step	5 th step	6 th step	7 th step	8 th step	9 th step	10 th step
0.046988	0.049076	0.052464	0.045713	0.044225	0.039196	0.034606	0.034379	0.033886	0.015108
	0.060005	0.061061	0.051227	0.04943	0.043519	0.038332	0.037725	0.037789	0.016825
		0.071068	0.057406	0.055246	0.048318	0.04246	0.041396	0.042142	0.018736
			0.064331	0.061747	0.053647	0.047032	0.045425	0.046996	0.020865
				0.069014	0.059564	0.052096	0.049846	0.052409	0.023236
					0.066133	0.057706	0.054697	0.058446	0.025876
						0.063919	0.06002	0.065177	0.028816
							0.065861	0.072685	0.03209
								0.081057	0.035736
									0.039796

Short rate volatilities implied by the BDT model

1 st step	2 nd step	3 rd step	4 th step	5 th step	6 th step	7 th step	8 th step	9 th step	10 th step
0.000136	0.10052	0.075875	0.056945	0.055626	0.052309	0.051134	0.046435	0.054508	0.053807

The short rates were used to compute the values for the model prices at each point in the development of the binomial tree. The

BDT bond prices were computed as an average of all the prices at a certain period in the tree.

61.629	66.227	70.871	75.722	80.13	84.426	88.312	91.783	95.202	98.512	100
	62.822	68.083	73.455	78.238	82.921	87.158	90.953	94.675	98.345	100
		65.101	71.025	76.198	81.29	85.901	90.044	94.093	98.161	100

			68.43	74.006	79.527	84.534	89.052	93.451	97.956	100
				71.658	77.625	83.052	87.968	92.743	97.729	100
					75.581	81.447	86.788	91.964	97.478	100
						79.713	85.505	91.107	97.199	100
							84.112	90.167	96.891	100
								89.136	96.55	100
									96.173	100
										100

BDT bond prices at each point in the binomial

BDT estimated prices at each period

1 year to maturity	2 years to maturity	3 years to maturity	4 years to maturity	5 years to maturity	6 years to maturity	7 years to maturity	8 years to maturity	9 years to maturity	10 year to maturity
97.499	92.504	88.276	84.302	80.228	76.046	72.158	68.018	64.524	61.629

Computing the bond values

The first step in the computation of the price for a ten-year bond will consist in generating random paths of the term structure of short rates for the treasury bonds from the bdt binomial tree. We used 10 000 simulations from the binomial tree. A random path is found by taking random drawings from a binomial distribution (actually a uniform distribution that can provide only the values 1 and 2).

The next step deals with the drawing of the spread over this path for each class as a random drawing from the lognormal distribution using the mean implied from the table 4 in Barnhill & Maxwell (2003) as the difference between the implied forward rate for the a rating class and the implied forward rate for the closest less risky rating class; the standard deviation of this short rate is provided in the same table in the paper.

Table 4

Term structure by bond rating class and mean reversion and volatility of term structures by bond rating class

	Asset class								
	Trea-sury	Aaa	Aa	A	Baa	Ba	B	Caa	
<i>Term structure information: 12/31/98</i>									
Time to maturity	1	4.59%	4.96%	5.00%	5.17%	5.53%	7.41%	8.78%	12.00%
	5	4.39%	5.09%	5.31%	5.55%	6.00%	8.79%	10.43%	15.01%
	10	4.59%	5.43%	5.75%	6.00%	6.49%	9.66%	11.43%	21.00%
	15	4.89%	5.80%	6.18%	6.43%	6.95%	10.12%	11.99%	21.00%
<i>Term structure parameter estimates (empirically estimated from 1/93–12/98)</i>									
Mean-reversion rate		0.048	0.061	0.062	0.058	0.084	0.171	0.069	0.142 ^a
Std. dev. of the short interest rate		0.007	0.010	0.010	0.010	0.011	0.014	0.010	0.039 ^a
Std. dev. of the interest rate spread (e.g. Ba–Baa)		n.a.	0.002	0.002	0.001	0.002	0.011	0.011	0.034 ^a

The term structure is estimated from Standard & Poor's *CreditWeek* and the Lehman Brothers bond database. Mean-reversion rates and volatilities of the short rates are estimated empirically over the January 1993 to December 1998 time period.

^aFor Caa–C.

We will look at the differences between each 2 closed rating classes and use the simulated risk free short rates from the BDT in order to generate the spreads for each class. Our

purpose is to estimate the forwards for each class to estimate the price of a ten year bond one year from now, having a coupon of \$117 as the bond presented in Table 3 from the

Barnhill & Maxwell paper (actually the objective is to provide a table that looks like this table for each rating class using our BDT model instead of the Hull and White model presented in the paper). Thus, we use the forward rate from year 1 to year 10 to compute the mean of the spread between rating classes. For example, we imply by interpolation the 1 to 10 forward rate for both the Aaa bonds and the Treasury bond from the table and use the difference between the two as a mean of the spread between them. The standard deviation of the spread is provided in the paper. Then we solve a nonlinear system of 2 equations with 2 unknowns (the mean and the standard deviation of the $\log(\text{spread})$). If $\log(\text{spread}) \sim N(\mu, \sigma^2)$ then we are looking for the μ and σ such that

$$\text{Mean}(\text{spread}) = e^{\mu + \frac{\sigma^2}{2}}$$

$$\text{Var}(\text{spread}) = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1)$$

The mean and the standard deviation are then used in a simulation to compute the spreads for the short rates of each rating class using the same random numbers (epsilons) for all the rating classes. The first spread is the spread obtained from table as the difference between the one year rates for two neighbouring classes.

The short rate are then used for the computation of the forward rates that will provide the prices of the ten-year bond at one year from issuance.

We keep the values for the bonds for all the 10 000 simulated risk free short rates and compute the value of the bonds for all the rating classes.

The final results for the means and standard deviations are presented are the following:

MC mean	MC st dev
1496.5	120.41
1452.1	182.18
1383.7	277.29
1278	375.76
1058.6	433.35
882.82	437.46
471.11	439.93

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