The Solution to the Bühlmann - Straub Model in the case of a Homogeneous Credibility Estimators

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Original paper, which contains a credibility model, for determining the linearized homogeneous credibility premiums at contract level. The fact that it is based on complicated mathematics, involving conditional expectations, needs not bother the user more than it does when he applies statistical tools like SAS, GLIM. These techniques can be applied by anybody on his own field of endeavour, be it economics, medicine, or insurance. We give a rather explicit description of the input data for the Bühlmann – Straub model used, only to show that in practical situations there will always be enough data to apply credibility theory to a real insurance portfolio.
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Introduction

In this article we first give the Bühlmann – Straub model – see Section 1 -, which consists of a portfolio of non-life insurance contracts. In Section 1 we will give the assumptions of the Bühlmann – Straub model and the optimal linearized credibility premium is derived. In Section 2 we give unbiased estimators for the structure parameters, such that if the structure parameters in the optimal linearized credibility premium are replaced by these estimators, a homogeneous estimator results. In Section 3 we will show that this last estimator is in fact the optimal linearized homogeneous credibility estimator.

1. The Bühlmann – Straub model

For this model we look upon the portfolio as represented in Diagram 1. We consider a portfolio which can be subdivided in groups consisting of contracts with common risk parameter, as in Diagram 1.

Contracts			1……………………j…………………k
Structure variables			θ_j
Observable variables with associated weights	p	1	$X_{j1}(w_{j1})$
	e	2	$X_{j2}(w_{j2})$
	r	⋮	⋮ ⋮
	i	⋮	⋮ ⋮
	o	⋮	⋮ ⋮
	d	t	$X_{jt}(w_{jt})$

Diagram 1 Bühlmann – Straub model

Each contract j = $\overline{1,k}$ is the average of a group of w_{jr} contracts, where w_{jr} is the weight (size) of the group j at time r, with r = $\overline{1,t}$.

Remark:

These weights arise when the contracts are replaced by averages of identical contracts (with the same risk parameter), and the weight then represents the number of such contracts.

The model consists of the structural variables θ_j and the observable variables X_{jr}, where j = $\overline{1,k}$ and r = $\overline{1,t}$. So, the contract j consists of the set of variables:

$(\theta_j, \underline{X}_j^{'}) = \theta_j, X_{jr}, r = \overline{1,t}$,

where j = $\overline{1,k}$; the contract indexed j is a random vector consisting of a random structure parameter θ_j and observations $X_{j1}, X_{j2}, \ldots, X_{jt}$, see Diagram 1:

$(\theta_j, \underline{X}_j^{'}) = (\theta_j, X_{j1}, \ldots, X_{jt})$,

where j = $\overline{1,k}$. Of course the variables X_{jr} represent the average of w_{jr} contracts grouped together at time r, as follows:

$X_{jr} = \frac{1}{w_{jr}} \sum_{i=1}^{w_{jr}} X_{jr}^{(i)}$, r = $\overline{1,t}$ and j = $\overline{1,k}$. The Bühlmann – Straub assumptions can be formulated as:

(BS₁): the contracts j = $\overline{1,k}$ (the pairs, the couples $(\theta_j, \underline{X}_j^{'})$ with j = $\overline{1,k}$) are independent; moreover, for every contract j = $\overline{1,k}$ and for $\boldsymbol{\theta_j} = \theta_j$ fixed, the variables $X_{j1}, \ldots, X_{jt}$ are conditionally independent. The variables $\boldsymbol{\theta_1, \ldots, \theta_k}$ are identically distributed. The observations X_{jr}, j = $\overline{1,k}$, r = $\overline{1,t}$ have finite variance.

(BS₂): $E(X_{jr} \mid \theta_j) = \mu(\theta_j)$, j = $\overline{1,k}$, r = $\overline{1,t}$ (we assume that all contracts have common expectation of the claim size as a function $\mu(\cdot)$ of the risk parameter θ_j, where j = $\overline{1,k}$).

$Var(X_{jr} \mid \theta_j) = \sigma^2(\theta_j)/w_{jr}$, j = $\overline{1,k}$, r = $\overline{1,t}$, where all $w_{jr} > 0$ (apart from the weighting factor w, we assume that the variance is also the same function of the risk parameter),with $X_{jr}^{(i)}$, i = $\overline{1,w_{jr}}$, j = $\overline{1,k}$, r = $\overline{1,t}$ satisfying the hypotheses: (BS'₁) and (BS'₂), where:

(BS'₁): for every j = $\overline{1,k}$ anf for $\boldsymbol{\theta_j} = \theta_j$ fixed, the variables $X_{jr}^{(i)}$, i = $\overline{1,w_{jr}}$, r = $\overline{1,t}$ are conditionally independent and identically distributed. The variables $\theta_1, \ldots, \theta_k$ are identically distributed and the observations $X_{jr}^{(i)}$, i = $\overline{1,w_{jr}}$, r = $\overline{1,t}$, j = $\overline{1,k}$ have finite variance, and:

(BS'₂): $E(X_{jr}^{(i)} \mid \theta_j) = \mu(\theta_j)$, i = $\overline{1,w_{jr}}$, j = $\overline{1,k}$, r = $\overline{1,t}$

$Var(X_{jr}^{(i)} \mid \theta_j) = \sigma^2(\theta_j)$, i = $\overline{1,w_{jr}}$, j = $\overline{1,k}$, r = $\overline{1,t}$

Consequence of the hypothesis (BS₁):

$Cov(X_{jr}, X_{jq} \mid \theta_j) = 0$, j = $\overline{1,k}$, r,q = $\overline{1,t}$, r < q.

Observations:

1) $\mu(\theta_j)$ is the pure net risk premium of the contract j, with j = $\overline{1,k}$.

2) the Bühlmann – Straub assumptions express the common characteristics of the risk under consideration.

Now, we derive the optimal linearized non-homogeneous credibility estimator.

The optimal linearized credibility estimators non-homogeneous are given in the following theorem:

Theorem 1: (linearized non-homogeneous credibility estimator in the Bühlmann – Straub model)

Under the hypotheses (BS₁) and (BS₂) of the Bühlmann – Straub model, the following optimal linearized non-homogeneous credibility estimator for $\mu(\theta_j)$, for some fixed j, is obtained:

$$M_j^a = \hat{\mu}(\theta_j) = (1 - z_j)m + z_j M_j, \qquad (1),$$

where $M_j = X_{jw} = \sum_{q=1}^{t} \frac{w_{jq}}{w_{j\cdot}} X_{jq}$ denotes the individual estimator for $\mu(\theta_j)$, and the resulting credibility factor for contract j is given by:

$z_j = aw_{j\cdot} / (aw_{j\cdot} + s^2)$,

with $a = Var[\mu(\theta_j)]$, $s^2 = E[\sigma^2(\theta_j)]$, $m = E[\mu(\theta_j)]$ as usual, where $w_{j\cdot} = \sum_{q=1}^{t} w_{jq}$, j= $\overline{1,k}$.

To be able to use result (1), one still has to estimate the portfolio characteristics m, s^2, a. Some unbiased estimators are given in the following section.

2. Parameter estimation

The estimators obtained in the previous section contained unknown structure parameters (the credibility premium for this Bühlmann – Straub model involves three unknown parameters: m, s^2 and a). So, the expressions for these (pseudo-) estimators are no longer statistics. But since the contracts are embedded in a collective of identical contracts, all providing independent information on the structure distribution, it is possible to give unbiased estimators of these quantities, so we can replace the unknown structure parameters by estimates. In this section, we consider different contracts, each with the same structure parameters: m, s^2 and a, so we can estimate these quantities using the statistics of the different contracts. Some unbiased estimators for the structure parameters: m, s^2 and a, are given in the following theorem. So, we will provide some useful estimators for the structure parameters: m, s^2 and a in the following theorem:

Theorem 2: (parameter estimation in the Bühlmann – Straub model)

The estimators:

$$\hat{m} = M_0 = X_{zw} = \sum_{j=1}^{k} \frac{z_j}{z.} W_{jw} \text{ (where: z. } = \sum_{j=1}^{k} z_j \text{)} \quad (2)$$

$$\hat{s}^2 = \frac{1}{k(t-1)} \sum_{j,s} w_{js} (X_{js} - X_{jw})^2 \quad (3)$$

$$\hat{a} = w.. \left[\sum_j w_{j\cdot} (X_{jw} - X_{ww})^2 - (k-1)\hat{s}^2 \right] / (w_{..}^2 - \sum_j w_{j\cdot}^2) \quad (4)$$

(where: w.. = $\sum_{j=1}^{k} w_{j\cdot} = \sum_{j=1}^{k}\sum_{q=1}^{t} w_{jq}$, $X_{ww} = \sum_{j=1}^{k} \frac{w_{j\cdot}}{w..} X_{jw}$) are unbiased estimators of the corresponding structure parameters, i.e.

$E(\hat{m}) = m$, $E(\hat{s}^2) = s^2$, $E(\hat{a}) = a$.

Remark:

In case m in (2) is estimated by M_0, we obtain a homogeneous linear combination of all observable variables, giving an unbiased estimate of m. The following section shows that this happens to give the optimal unbiased homogeneous linearized credibility result.

3. Bühlmann – Straub model for homogeneous credibility estimators

Replacing the structure parameter m by an unbiased estimate results in a homogeneous credibility estimator. In Section 3, we will show that this last estimator is in fact the optimal linearized homogeneous credibility estimator.

Now, we derive the optimal linearized homogeneous credibility estimator.

The optimal linearized credibility estimators, homogeneous are given in the following theorem:

Theorem 3: (homogeneous credibility estimators in the Bühlmann – Straub model)

The solution of the following minimization problem:

$$\underset{c_j}{Min} E\left\{\left[\mu(\theta_j) - \sum_{i=1}^{k}\sum_{r=1}^{t} c_{jir} X_{ir}\right]^2\right\}, \quad (5)$$

($c_j=(c_{jir})_{i,r}$), such that $E[\mu(\theta_j)]=\sum_{i,r} c_{jir} E(X_{ir})$ (6),

is $M_j^a = (1 - z_j)M_0 + z_jM_j$ (7),

whit z_j as in Theorem 1.

Proof:

Let j be fixed. The unbiasedness restriction (6) can be written as

$$\sum_{i,r} c_{jir} = 1 \quad (8),$$

because $E(X_{ir}) = E[\mu(\theta_j)] = m$.

We insert it in the expectation in (5), and add it to the function to be optimized with a Lagrange multiplier $2\alpha / m$. The following problem results:

$$\underset{c_{j,\alpha}}{Min}\left(E\left\{\left[\mu(\theta_j) - m - \sum_{i,r} c_{jir}(X_{ir} - m)\right]^2\right\} + 2\alpha\left(1 - \sum_{i,r} c_{jir}\right)\right) \quad (9)$$

Since (9) is the minimum of a positive definite quadratic form, it suffices to find a solution with all partial derivatives equal to zero. Taking the derivative with respect to $c_{ji'r'}$ gives for $i' = \overline{1,k}$, $r' = \overline{1,t}$:

$$\alpha + Cov[\mu(\theta_j), X_{i'r'}] = \sum_{i,r} c_{jir} \, Cov(X_{ir}, X_{i'r'})$$

$$\alpha + \delta_{ji'}a = \sum_{i,r} c_{ji'r} \, (a + \delta_{rr'}s^2 / w_{i'r}), \; i' = \overline{1,k}, \; r' = \overline{1,t}$$

$(Cov[\mu(\theta_j), X_{i'r'}] = \delta_{ji'}a$, $Cov(X_{i'r}, X_{i'r'}) = a + \delta_{rr'}s^2/w_{i'r}$, $Cov(X_{ir}, X_{i'r'}) = 0$ if $i \neq i'$).

These equations can be simplified as follows:

$$\alpha + \delta_{ji'}a = ac_{ji'.} + s^2 c_{ji'r'} / w_{i'r'} \qquad (10),$$

where $c_{ji'.} = \sum_{i,r} c_{ji'r}$. Multiplying each equation with $w_{i'r'}$ and summing these equations over the index r', gives for each i':

$(\alpha + \delta_{ji'}a)\, w_{i'.} = c_{ji'.}aw_{i'.} + s^2 c_{ji'.}$

So, $c_{ji'.} = (\alpha + \delta_{ji'}a)\, w_{i'.} / (s^2 + aw_{i'.})$ (11).

Inserting (11) into (10) gives an expression for $c_{ji'r'}$:

$c_{ji'r'} = (\alpha + \delta_{ji'}a)\,(1 - aw_{i'.} /(aw_{i'.} + s^2))\, w_{i'r'}/s^2$

$= (\alpha + \delta_{ji'}a)\,(1 - z_{i'})w_{i'r'}/s^2$

From this the estimator (7) for $\mu(\theta_j)$ becomes:

$$\hat{\mu}(\theta_j) = \sum_{i'r'} (\alpha + \delta_{ji'}a)(1 - z_{i'}) w_{i'r'}/s^2 X_{i'r'},$$

where still α has to be determined in such a way that (6) holds, too. Summing all the $c_{ji'.}$ of (11), one gets:

$$1 = \sum_{i'} c_{ji'.} = \alpha \sum_{i'} z_{i'} / a + z_j = \alpha z./a + z_j$$

and the resulting value for $\alpha = a(1 - z_j) / z.$, inserted in (12), gives after some algebraic manipulations the following optimal estimator for $\mu(\theta_j)$:

$$M_j^a = \hat{\mu}(\theta_j) = (1 - z_j)\, X_{zw} + z_j X_{jw} \qquad (12)$$

(for more details, see [1], from References).

So, the theorem is proven.

Remark:

Finally we want to remark that in case one uses the formula:

$$M_j^a = (1 - \hat{z}_j)\, M_0 + \hat{z}_j M_j$$

we have $E(M_j^a) \neq m$ in case the estimators from Theorem 2 are used, because then $\hat{z}_j$ is dependent of M_j and M_0.

Of course the attractive property of unbiasedness is lost this way, but we can still expect the resulting estimators to be good. For instance when an estimator is a maximum likelihood estimator for a parameter, so are functions of it for these functions of the parameter.

Conclusions

One likely choice in the minimization problem:

$\underset{g(\cdot)}{Min}\, E\{[\mu(\theta_j) - g(X_{j1},..., X_{jt})]^2\}$, giving easily computable premiums, is:

$g(X_{j1}, \ldots, X_{jt}) = c_0 + \sum_{i=1}^{k}\sum_{r=1}^{t} c_{jir} X_{ir}$, leading to so-called linearized credibility results.

Another possibility is to limit oneself to unbiased homogeneous linear estimators, by requiring additionally $c_0 = 0$ and:

$$E[\mu(\theta_j)] = \sum_{i,r} c_{jir} \, E(X_{ir})$$

Proceeding this way one gets homogeneous linear credibility formulae. By the requirement of unbiasedness the sum of the credibility premiums equals the global premium on the top-level.

In this paper we demonstrated that the estimators obtained for the pure net risk premium on contract level are the best linearized credibility estimators, homogeneous for the Bühlmann – Straub model, using the greatest accuracy theory.

So, the article provides the means to calculate the credibility premiums at contract level, which represents the most recent developments in Bayesian credibility theory. They certainly present the only solution where insurance industry faces risks with basic risk characteristics that cannot be assigned to any established collective or with coverage under circumstances not earlier met.

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