An Implementation of Infinitesimals Using Mathematica®

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The paper presents an axiomatic approach of Nonstandard Analysis due to V. Benci and M. di Nasso. The implementation in Mathematica® of some primitives which handle hyperreals is the purpose of a distinct section. A way of using Nonstandard Analysis in solving economic problems can be found in H. Varian, but this is not the goal of the present paper.
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Introduction

The mathematical modeling of any phenomenon (including the economic ones) is based on the functionality of the principle of commensurability of various classes of parameters. There are stipulated the following: a priori existence of the set of integer numbers; existence of a method of measurement specific for a certain class of parameters; existence of the representation of the parameters' properties through the measurement previously made.

The notion of number is largely used, but up to now its complexity has not been completely revealed. Any high schooler operates with numbers from sets of natural, integer, rational, real or complex numbers. □ is the ordered sub-field contained in any ordered field. Another important property of □ is related to the quality of Archimedean ordered field:

If $a > 0$ and $b > 0$, then there is n natural so that $na > b$.

□ is also Archimedean ordered field which, in addition, enjoys the property of completeness: any nonempty part S of □ upper bounded admits a least upper bound. It is known that any two complete ordered fields are isomorphe with respect to the following operations: addition, multiplication and ordering. □ is not an ordered field, but there are ordered fields which contain the set □ as an own sub-ordered field. Any ordered field $□^*$ which contains □ as an own sub-ordered field is not Archimedean and, consequently, it is not complete.

The notions of "small infinite" and "large infinite" were used by the authors of differential calculus: Leibniz and Newton. Partial removed by the $\varepsilon - \delta$ language of A.Cauchy, K. Weierstrass and B.Riemann, criticised by Bertrand Russell at the beginning of the 20th century as inutile, wrong and contradictory, the notions revert to the rigour requested by the constructions of mathematical analysis used in the work of Abraham Robinson with regard to Nonstandard Analysis in the 7th decade of the last century.The following Robinson theorem presents the general framework for the below results.

Theorem (A. Robinson). There is a set $□^*$ having the following properties:

1. □ is a strict subset of $□^*$.
2. To every $f : □^n \to □$, there corresponds $f^* : (□^*)^n \to □^*$, so that $f = f^*$ over $□^n$.
3. To every n-ary relation ρ in □ $(n \geq 1)$, there corresponds a n-ary relation ρ^* in $□^*$ so that ρ coincides with ρ^* over □. The equality relation in □ corresponds to the equality relation in $□^*$.
4. Any statement δ phrased in terms of: (i) some real numbers (*fixed*); (ii) some real functions (*fixed*); (iii) some relations in □ (*fixed*); (iv) some variables with values in □; (v) some quantifiers and logical operations, is true with respect to □ iff the statement δ^* is true with respect to $□^*$, where δ^* is obtained from δ by replacing each function $f(x_1, \ldots, x_n)$ with the corresponding function $f^*(x_1, \ldots, x_n)$ and each relation $\rho(x_1, \ldots, x_n)$ with the corresponding relation $\rho^*(x_1, \ldots, x_n)$ and by extending the variables from □ to $□^*$.

We considered $n \geq 1$.
An element $a \in □^*$ is called infinitesimal if $|a| < r, \quad \forall r \in □,\ r > 0$. A number $a \in □^*$ is called finite if there is $r \in □$ so that $|a| \leq r$. A number $a \in □^*$ is called infinite if $|a| > r, \quad \forall r \in □$.

It could be said that the existence of infinitesimals contradicts the "American dream" – if someone is born poor (an infinitesimal *a*), then even if they work hard to improve their condition and become *a*+*a* or *a*+*a*+*a* or *na* etc., they will always remain poor (infinitesimal).
In the opinion of A. Robinson, the Nonstandard Analysis rather consists in developping new inferring methods and not in introducing new mathematical entities; within the Nonstandard Analysis, infinite numbers and infinitely small numbers do not have a weaker or a more powerful reality than that of irrational numbers in Standard Analysis. As the irrational numbers, the Nonstandard (or ideal) ones are introduced through infinite processes. The elements in $□^*$ appear as classes of equivalent sequences of rational numbers (with the proper difference in defining the two equivalence relations).

2. An elementary axiomatics

Many attempts have been made in order to simplify the foundational matters and give an easy presentation (the formalism of Robinson's original presentation appeared too technical and not directly usable by those mathematicians without a good background in logic). Most notably, the pioneering work by W.A.J. Luxemburg, the *superstructure approach* presented by A. Robinson jointly with E. Zakon; the elementary axiomatics given by H.J. Keisler, the algebraic presentation of hyperreals by W.S. Hatcher, and finally the introduction by W. Henson. The recent new presentation of nonstandard analysis, that is named the *Alpha-Theory*, shows that technical notions such as superstructure, ultrafilter, ultrapower, bounded formula and the transfer principle, are not needed to rigorously develop calculus with infinitesimals.
V.Benci and M.di Nasso give the following indication for a rigorous definition of the Alpha-Theory. Let $L = \{\in, A, J\}$, where A is a set of atoms and J a binary relation.

J1. *Extension Axiom.* If φ is a sequence, then there exists a unique x such that $J(\varphi, x)$. Vice versa, if $J(\varphi, x)$ holds for some x, then φ is a sequence.

J2. *Composition Axiom.* If φ and ψ are sequences and if f is any function such that compositions $f \circ \varphi$ and $f \circ \psi$ make sense, then

$$\forall x[(J(\varphi, x) \wedge J(\psi, x)) \rightarrow \exists y[J(f \circ \varphi, y) \wedge J(f \circ \psi, y)]]$$

J3. *Number Axiom.* Let $r \in □ \subset A$. If $c_r : n \mapsto r$ is the constant sequence with value r, then: $\forall x[J(c_r, x) \rightarrow x = r]$. If $1_{\mathbf{N}} : n \mapsto n$ is the identity sequence on □, then $\forall x[J(1_{\mathbf{N}}, x) \rightarrow x \notin \mathbf{N}]$.

J4. *Pair Axiom.* For all sequences φ, ψ and ϑ such that $\vartheta(n) = \{\varphi(n), \psi(n)\}$ for all n:

$$\forall x \forall y \forall z[(J(\varphi, x) \wedge J(\psi, y) \wedge J(\vartheta, z)) \rightarrow z = \{x, y\}]$$

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J5. *Internal Set Axiom.* If ψ is a sequence of atoms, then $\forall x[J(\varphi, x) \rightarrow x \in A]$. If $c_\varnothing$ is the constant sequence with value the empty set, then $J(c_\varnothing, \varnothing)$ is true. If φ is a sequence of nonempty sets, then:

$$\forall x[J(\varphi, x) \rightarrow \forall y[y \in x \leftrightarrow \exists \psi[\psi \in \varphi \wedge J(\psi, y)]]].$$

They use the notation $\psi \in \varphi$ as a shorthand for the following: ψ and φ are sequences and $\psi(n) \in \varphi(n)$ for all $n \in □$.

Definition. The *Alpha-Theory* is the first-order theory in the language $L = \{\in, A, J\}$, where $\in$ and J are binary relation symbols and A is a constant symbol, and whose set of axioms consists of:

- All axioms of Zermelo-Fraenkel set theory with atoms (ZFCA), with the only exception of the axiom of foundation. The separation and replacement schemas are also considered for formulas containing the J-symbol.
- The five axioms **J1**, . . . , **J5** as given above.

Definition. The set of hyperreal numbers is $\mathbf{R}^* = \{x \mid J(\varphi, x) \wedge \varphi : \mathbf{N} \rightarrow \mathbf{R}\}$

Definition. Given a hyperreal number , the monad of ξ is the set

$$mon(\xi) = \{x \in \mathbf{R}^* \mid x - \xi \text{ infinitesimal } \}.$$

The galaxy of ξ is the set $gal(\xi) = \{x \in \mathbf{R}^* | x - \xi \text{ is finite}\}$.

Thus *mon*(0) is the set of infinitesimal and *gal*(0) is the set of bounded numbers. All the real numbers are contained in *gal*(0); the unique real number contained in *mon*(0) is "0". Any two monads are equal or disjoint and any two galaxies are equal or disjoint. In addition,

$$mon(\xi) = \{\xi + t \mid t \in mon(0)\},$$
$$gal(\xi) = \{\xi + t \mid t \in gal(0)\}.$$

Theorem (Shadow Theorem). Every finite hyperreal number ξ is infinitely close to a unique real number *r*, called the shadow of ξ. Symbolically, $r = sh(\xi)$.

3. A Mathematica® Implementation

The goal of this section is an instrumental one: we propose a framework for hyperreal numbers. The proposed implementation tries to use only a part of these numbers that is sufficient for explaining the distinction between various infinitesimals, as well as for explaining the distinction between various infinites. The instructions come to complete the package proposed by P.A. Rubin and taken over by H.R. Varian.

```
IsHyperreal[number_]:=And[MemberQ[{Function},Head[number]],
                     Length[Complement[Variables[number[e]],{e}]]<1];
MakeHyperreal[number:(_?IsHyperreal|_?NumericQ)]:=
                     If[IsHyperreal[number],number,
                     If[And[Not[MemberQ[{Symbol},Head[number]]],
                         Simplify[number\[Element]Reals]],
                     Function[e,number]]]
IsBound[hyperreal_?IsHyperreal]:=Module[{t},Off[Power::infy];
                     t=hyperreal[0];On[Power::infy];
                     Not[MemberQ[{Infinity,-Infinity,ComplexInfinity},t]]];
IsBound[hyperreal_]:=False;
UpperBoundOfOrderScreening=300;
DominantOrder[hyperreal_?IsHyperreal]:=Module[{t={hyperreal,0}},
                     If[IsBound[hyperreal],
                     While[And[t[[2]]>-UpperBoundOfOrderScreening,
                         t[[1]][0]==0],t={Function[e,
                         Evaluate[D[#1[e],e]/(1-#2)]], -1+#2}& @@t];
                     If[And[t[[1]][0]==0,
                         t[[2]]==-UpperBoundOfOrderScreening],t[[2]]=0],
                     Off[Power::infy];
                     While[And[t[[2]]<UpperBoundOfOrderScreening,
                         MemberQ[{Infinity,-Infinity,ComplexInfinity},
                           t[[1]][0]]], t={Function[e,Evaluate[e #1[e]]],
                           1+#2}& @@ t]; On[Power::infy]];
                     {t[[1]][0],t[[2]]}];
UnboundFrame[hyperreal_?IsHyperreal]:=Module[{n},
                     n=Evaluate[DominantOrder[hyperreal]][[2]];If[n>0,
                     Function[e,Evaluate[Normal[Series[e^n hyperreal[e],
                           {e,0,n-1}]]/e^n]],
                     Function[e, 0]] ];
CurrentFrame[hyperreal_?IsHyperreal]:=Module[{hyp},
                     hyp=Function[e,Evaluate[hyperreal[e]
                           -UnboundFrame[hyperreal][e]]];
                     MakeHyperreal[hyp[0]] ];
InfinitesimalFrame[hyperreal_?IsHyperreal]:=Function[e,Evaluate[
                     hyperreal[e]-UnboundFrame[hyperreal][e]-
                           CurrentFrame[hyperreal][e]]];
IsReal[hyperreal_?IsHyperreal]:=!MemberQ[Variables[hyperreal[e000]],e000];
IsReal[hyperreal_]:=False;
Unprotect[Positive,Plus,Times,Power];
```

```
Positive[hyperreal_?IsHyperreal]:=DominantOrder[hyperreal][[1]]>0;
Plus[h1_?IsHyperreal,h2_?IsHyperreal]:=Function[e,Evaluate[h1[e]+h2[e]]];
Times[h1_?IsHyperreal,h2_?IsHyperreal]:=
                     Function[e,Evaluate[ExpandAll[h1[e]*h2[e]]]];
Power[h1_?Positive,h2_?IsHyperreal]:=
                     Function[e,Evaluate[Power[h1[e],h2[e]]]];
Unprotect[Positive,Plus,Times,Power];
Shadow[hyperreal_?IsBound]:=CurrentFrame[hyperreal][0];
Infinitesimal[hyperreal_?IsBound]:=
                     If[Positive[InfinitesimalFrame[hyperreal]],
                         IA[Shadow[hyperreal]],
                     If[Positive[InfinitesimalFrame[-hyperreal]],
                         IB[Shadow[hyperreal]],
                     Shadow[hyperreal]]];
```

We mention that a hyperreal number is a λ-expression (by analogy with LISP) and the implementation is not exhaustive with respect to the operations made with these numbers.

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