Principal Component Analysis with Applications in Image Restoration

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The research work reported in the paper aims to propose a new image reconstruction method based on the features extracted from the noise given by the principal components of the noise covariance matrix. We consider the additive normal distributed degradation model and we use a code shrinkage technique to remove noise from the white vector resulted using the principal component representation. Classical feature extraction and data projection methods have been extensively investigated in the pattern recognition and exploratory data analysis literature. During the past decade a large number of artificial neural networks and learning algorithms have been proposed for solving feature extraction problems, most of them being adaptive in nature and well-suited for many real environments where adaptive approach is required. Principal Component Analysis, also called Karhunen-Loeve transform is a well-known statistical method for feature extraction, data compression and multivariate data projection and so far it has been broadly used in a large series of signal and image processing, pattern recognition and data analysis applications.
Keywords*: feature extraction, pattern recognition, PCA, image processing, noise removal, shrinkage function, sparse code shrinkage methods*

1 On a certain class of PCA learning schemes

Principal Component Analysis, also called Karhunen-Loeve transform is a well-known statistical method for feature extraction, data compression and multivariate data projection and so far it has been broadly used in a large series of signal and image processing, pattern recognition and data analysis applications. A large number of specialized neural networks and learning algorithms have been proposed to perform principal component analysis (PCA) tasks.One of the most frequently used method in the study of the convergence properties corresponding to different stochastic learning PCA algorithms, is derived from Kushner and Clark [4] developments and basically proceeds by reducing the problem to the analysis of asymptotic stability of the trajectories of a dynamic system whose evolution is described in terms of the ODE. The Generalized Hebbian Algorithm (GHA) extends the Oja's learning rule for learning the first principal components, the extension being essentially based on the Hotteling deflation technique.

Let us assume that the first *m* principal components have to be encoded as local memories (W_j) of a neural system, where $1 \le m \le n$ and *n* is the dimension of the input data. The computing layer consists of *m* units of local memories (W_j), each neuron being interconnected with all neurons of rank greater or equal then its rank.

The input is sampled from a *n*-dimensional stochastic process $X(t)=[X^{(1)}(t), \ \ldots, \ X^{(n)}(t)]$ stationary in the large sense, $E(X(t))=0$. At any moment *t*, each neuron *j*, $j \ge 1$, receives two inputs, the original signal $X(t)$ and the deflated signal $X_j(t)$ and computes two outputs,

(1) $Y_j(t)=W_j^T(t)X(t)$

(2) $\tilde{Y}_j(t)=W_j^T(t)\tilde{X}_j(t)$

where $\tilde{X}_j(t)=\tilde{X}_{j-1}(t)-Y_{j-1}(t)W_{j-1}(t)$, $j \ge 2$.

The updating of the local memories is given by,

(3) $W_1(t+1)=W_1(t)+\eta(t)\left[Y_1(t)X(t)-Y_1^2(t)W_1(t)\right]$

(4) $W_j(t+1)=W_j(t)+\eta(t)\left[\tilde{Y}_j(t)\tilde{X}_j(t)-\tilde{Y}_j^2(t)W_j(t)\right]$

for $j \ge 2$.

The variant of the GHA proposed by Sanger [10] simplifies the learning scheme by using only the output $Y_j(t)$ for both, updating the synaptic memories and signal deflation. According to the Sanger PCA learning algorithm, the updating is given by,

(5) $W_j(t+1) = W_j(t) + \eta(t)\left[Y_j(t)X(t) - Y_j(t)\sum_{i=1}^{j} Y_i(t)W_i(t)\right]$

for $j \geq 1$.

According to the Kushner construction, the ODE describing the dynamics is,

(6) $\frac{dW_1(t)}{dt} = SW_1(t) - \left(W_1^T(t)SW_1(t)\right)W_1(t)$

(7) $\frac{dW_j(t)}{dt} = SW_j(t) - \left(W_j^T(t)SW_j(t)\right)W_j(t) - - \sum_{i=1}^{j-1}\left(W_i^T(t)SW_j(t)\right)W_i(t)$

for $j \geq 2$, where $S = E\left(X(t)X^T(t)\right)$.

The APEX learning algorithm proposed by Kung and Diamantaras [3] generalizes the idea of lateral influences by imposing a certain learning process to the weights of lateral connections. The output of each neuron *j*, $j \geq 2$, is computed from its own output and the effects of the outputs corresponding to all neurons *i*, $1 \leq i \leq j-1$, weighted by the coefficients $a_{ij}(t)$,

(8) $Y_j(t) = W_j^T(t)X(t) - \sum_{i=1}^{j-1} a_{ij}(t)Y_i(t)$

The learning scheme for the local memories is essentially the Oja's learning rule taken for the transformed outputs Y_j,

(9)
$W_j(t+1) = W_j(t) + \eta(t)\left[Y_j(t)X(t) - Y_j^2(t)W_j(t)\right]$

The learning scheme for the weights of lateral connections is given by,

(10) $a_{ij}(t+1) = a_{ij}(t) + \eta(t)\left[Y_i(t)Y_j(t) - Y_j^2(t)a_{ij}(t)\right]$

2. The new approach of noise removal in image processing

The ultimate goal of restoration techniques is to improve an image in some sense. The restoration can be viewed as a process that attempts to reconstruct or recover an image that has been degraded by using some *a priori* knowledge about the degradation phenomenon. Thus restoration techniques are oriented toward modeling the degradation and applying the inverse process in order to recover the original image. This approach usually involves formulating a criterion of goodness that will yield some optimal estimate of the desired result. The effectiveness of restoration techniques mainly depends on the accuracy of the image modeling.

The new technique proposed in this paper is based on process the white vector obtain using the PCA representation of the input signal. The model is described in the follows.

Let $\mathbf{X} = \mathbf{X}_0 + \boldsymbol{\eta}$ the observed images, where

- $\mathbf{X}_0$ is the original image set, distributed $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
- $\boldsymbol{\eta}$ is the noise component, distributed $\mathrm{N}\left(\mathbf{0}, \sigma^2\mathbf{I}_n\right)$.

The observed data represents a set of monochrome images, processed using a 8×8 block decomposition and liniarization $(n = 64)$. The preprocessing task consists in normalization and then centering the data. We obtain,

(11) $\mathbf{Y} = \mathbf{X} - \mathrm{E}(\mathbf{X}) = \mathbf{X}_0 - \boldsymbol{\mu} + \boldsymbol{\eta}$.

Let us assume that $0 < \sigma^2 < 1$ (it results by using the normalization preprocessed data).

The covariance matrix of the resulted data is, $\mathrm{Cov}\left(\mathbf{Y}, \mathbf{Y}^T\right) = \boldsymbol{\Sigma} + \sigma^2\mathbf{I}_n$.

Let $\boldsymbol{\Phi}_0$ be the ortonormal matrix of eigen vectors of $\boldsymbol{\Sigma}$ and $\boldsymbol{\Lambda}_0 = \mathrm{diag}\left\{\lambda^0{}_1, \lambda^0{}_2, ..., \lambda^0{}_n\right\}$ the diagonal matrix of their corresponding eigen values. Let $\boldsymbol{\Phi}$ be the ortonormal matrix of eigen vectors of $\boldsymbol{\Sigma} + \sigma^2\mathbf{I}_n$ and $\boldsymbol{\Lambda} = \mathrm{diag}\left\{\lambda_1, \lambda_2, ..., \lambda_n\right\}$ the diagonal matrix of their corresponding eigen values. We get,

(12) $\boldsymbol{\Phi} = \boldsymbol{\Phi}_0$

and, for $1 \leq i \leq n$,

(13) $\lambda_i = \lambda^0{}_i + \sigma^2$

The resulted transform is described as follows,

(14) $\mathbf{Z} = \boldsymbol{\Phi}\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Phi}^T\mathbf{Y} = \boldsymbol{\Phi}\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Phi}^T\left(\mathbf{X}_0 - \boldsymbol{\mu}\right) + \boldsymbol{\Phi}\boldsymbol{\Lambda}^{-\frac{1}{2}}\boldsymbol{\Phi}^T\boldsymbol{\eta}$.

In order to extract the noise component in (14), the II degree approximation of $\boldsymbol{\Lambda}^{-\frac{1}{2}}$ in terms of $\boldsymbol{\Lambda}_0^{-\frac{1}{2}}$ is computed. For all $1 \leq i \leq n$,

$\frac{1}{\sqrt{\lambda_i}} = \frac{1}{\sqrt{\lambda^0{}_i + \sigma^2}}$, and, using the function $\mathrm{f}(x) = \frac{1}{\sqrt{\lambda^0{}_i + x^2}}$, we obtain $\mathbf{\Lambda}^{-\frac{1}{2}} \cong \mathbf{\Lambda}_0^{-\frac{1}{2}} - \frac{1}{2}\sigma^4 \mathbf{\Lambda}_0^{-\frac{3}{2}}$

Using the above mentioned approximation, we obtain the following representation of the input signal (14),

$$(15)\ \mathbf{Z} = \mathbf{\Phi}\left(\mathbf{\Lambda}_0^{-\frac{1}{2}} - \frac{1}{2}\sigma^4 \mathbf{\Lambda}_0^{-\frac{3}{2}}\right)\mathbf{\Phi}^T(\mathbf{X}_0 - \boldsymbol{\mu}) + \mathbf{\Phi}\left(\mathbf{\Lambda}_0^{-\frac{1}{2}} - \frac{1}{2}\sigma^4 \mathbf{\Lambda}_0^{-\frac{3}{2}}\right)\mathbf{\Phi}^T \boldsymbol{\eta} =$$

$$= \mathbf{\Phi}\mathbf{\Lambda}_0^{-\frac{1}{2}}\mathbf{\Phi}^T(\mathbf{X}_0 - \boldsymbol{\mu}) + \mathbf{\Phi}\mathbf{\Lambda}_0^{-\frac{1}{2}}\mathbf{\Phi}^T\boldsymbol{\eta} - \frac{1}{2}\sigma^4 \mathbf{\Phi}\mathbf{\Lambda}_0^{-\frac{3}{2}}\mathbf{\Phi}^T(\mathbf{X}_0 - \boldsymbol{\mu} + \boldsymbol{\eta}).$$

The term $\mathbf{Z}_0 = \mathbf{\Phi}\mathbf{\Lambda}_0^{-\frac{1}{2}}\mathbf{\Phi}^T(\mathbf{X}_0 - \boldsymbol{\mu})$ in (15) is noise free, because $\mathbf{\Phi} = \mathbf{\Phi}_0$. We consider in the follows that the term $\frac{1}{2}\sigma^4 \mathbf{\Phi}\mathbf{\Lambda}_0^{-\frac{3}{2}}\mathbf{\Phi}^T(\mathbf{X}_0 - \boldsymbol{\mu} + \boldsymbol{\eta})$ in (15) can be neglected, so that we can use for noise removal only the term,

$$(16)\ \mathbf{t} = \mathbf{\Phi}\mathbf{\Lambda}_0^{-\frac{1}{2}}\mathbf{\Phi}^T(\mathbf{X}_0 - \boldsymbol{\mu}) + \mathbf{\Phi}\mathbf{\Lambda}_0^{-\frac{1}{2}}\mathbf{\Phi}^T\boldsymbol{\eta}.$$

We consider the transform, $\boldsymbol{\eta}' = \mathbf{\Phi}\mathbf{\Lambda}_0^{-\frac{1}{2}}\mathbf{\Phi}^T\boldsymbol{\eta}$. Using the fact that $\mathrm{Cov}(\boldsymbol{\eta}', \boldsymbol{\eta}'^T) = \sigma^2\mathbf{\Sigma}^{-1}$, the transformed vector defined by

$$(17)\ T(\mathbf{t}) = \mathbf{\Phi}\mathbf{\Lambda}_0^{\frac{1}{2}}\mathbf{\Phi}^T\mathbf{t} = \mathbf{\Phi}\mathbf{\Lambda}_0^{\frac{1}{2}}\mathbf{\Phi}^T\mathbf{Z}_0 + \mathbf{\Phi}\mathbf{\Lambda}_0^{\frac{1}{2}}\mathbf{\Phi}^T\boldsymbol{\eta}' = \mathbf{Z}_0' + \boldsymbol{\eta}''$$

is so that the noise component $\boldsymbol{\eta}''$ is $\mathrm{N}(\mathbf{0}, \sigma^2\mathbf{I}_n)$.

The noise removal procedure applied on the vector defined by (17) is then defined by a shrinkage function. A several examples of shrinkage function are describe in the follows [6].

1. $g(u) \cong \mathrm{sign}(u)\max\left(0, |u| - \sqrt{2}\sigma^2\right)$.
2. (50) $g(u) = \frac{1}{1+\sigma^2 a}\mathrm{sign}(u)\max\left(0, |u| - b\sigma^2\right)$
3. $g(u) = \mathrm{sign}(u)\max\left(0, \frac{|u| - ad}{2} + \frac{1}{2}\sqrt{(|u| + ad)^2 - 4\sigma^2(\alpha+3)}\right)$,

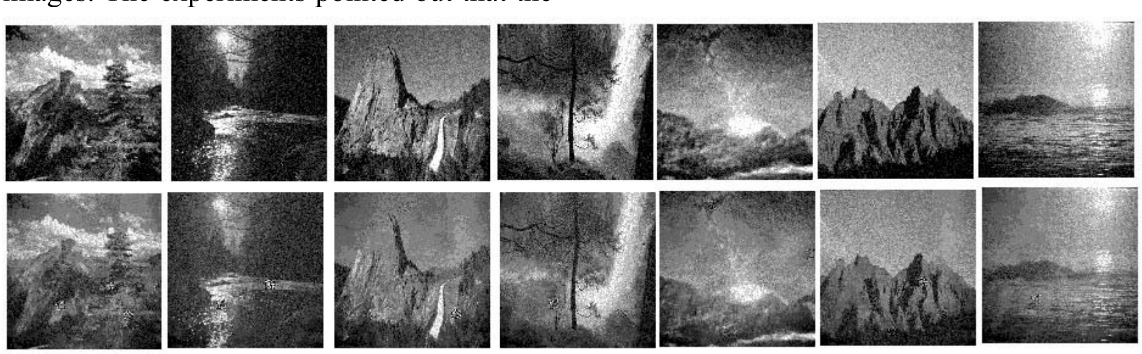
if $(|u| + ad)^2 - 4\sigma^2(\alpha+3) \geq 0$, where $a = \sqrt{\frac{\alpha(\alpha+1)}{2}}$, else $g(u) = 0$.

3. Experimental report on using the proposed algorithm in gray level images analysis

The tests on the efficiency of the proposed algorithm were performed on the gray level images. The experiments pointed out that the good quality of our restoration method. We consider for experiment several type of images. The set of observed data consists of image with similar statistical properties.

In the following we present a result of applying the above mentioned algorithm.

The set of noisy samples / The resulted images



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