Internet congestion control model with feedback delay

Lecturer PhD. Gabriela MIRCEA
Department of Economic Informatics
Faculty of Economic Sciences, West University of Timisoara
Professor PhD. Dumitru OPRIS
Department of Applied Mathematics
Faculty of Mathematics, West University of Timisoara

In this paper, we consider an Internet model with n access links, which respond to congestion signals from the network, and study the bifurcation of such a system. By choosing the gain parameter as a bifurcation parameter, we prove that a Hopf bifurcation occurs. The stability of bifurcating periodic solutions and the direction of the Hopf bifurcation are determined by applying the normal form theory and the center manifold theorem. Finally a numerical example is given to verify the theoretical analysis.
Keywords: *Internet model, feedback delay, Hopf bifurcation.*

Introduction

Congestion control mechanisms and active queue management schemes (AQM) for Internet have extensively studied. In [3], a stability condition was provided for a single proportionally fair congestion controller with delayed feedback.

In this paper, we consider an Internet model with n $(n \geq 3)$ link and single source, which can be formulated as a

$$\dot{x}_i(t) = k(w - a x_i(t-t)\, p(x_i(t-t) - b \sum_{\substack{j=1 \\ j \neq i}}^{n} x_j(t-t)\, p(x_i(t-t))), i = 1,..., n \qquad \textbf{(1)}$$

where $x_i(t)$ is the sending rate of the source i at time t, k is a positive gain parameter, t is the sum of forward and return delays, w is a target (set –point), and the congestion indication function $p(\cdot)$ is increasing, non-negative, and not identically zero, which can be viewed as the probability that a packet at the source receives a "mark"- a feedback congestion indication signal.

According to the existing theoretical results [2] on stability analysis of system (1), the rate control algorithm cannot always guarantee a stationary sending rate. We will show in this paper that a Hopf bifurcation will occur when the positive gain parameter k passes through a critical value, i.e, a family of periodic solutions will bifurcate out from the equilibrium. On the one hand, the bifurcations, which involve emergence of oscillatory behaviors, may provide an explanation for the parameter sensitivity observed in practice; and on the other hand, if we understand more about the bifurcation behaviors of the Internet congestion control systems, we can apply the existing effective bifurcation control methods to achieve some desirable system behaviors that benefit congestion controls. The rest of the paper is organized as follows. In section 2, we study the existence of the Hopf bifurcation. Based on the normal form theory and the center manifold theorem introduced in [2], the formulas for the determining the properties of the Hopf bifurcating periodic solutions are derived in section 3. In section 4, a numerical example is given to demonstrate the theoretical analysis. Finally, conclusions are drawn in section 5.

Existence of the Hopf bifurcation in the Internet model

Without loss of generality, we assume that the function $f(x) = xp(x)$ is a nonlinear function and its third –order continuous derivative exists, and that the delayed differential equations (1) is supplemented with an

initial conditions of the form:
$x(q) = j(q), q \in [-t, 0]$, **(2)**, where
$x(q) = (x_1(q),..., x_n(q))^T$, $j(q) = (j_1(q),..., j_n(q))^T$,
and $j_i(\cdot), i = 1,..., n$ is assumed to be a real – valued function continuous on $[-t, 0]$. The symmetric equilibrium point of (1) is $X^* = (x^*,..., x^*)^T$, where x^* satisfies:

$w = (a + (n-1)b) f(x^*)$ **(3)**

The linearized equation of (1) in x^* is:

$\dot{u}(t) = -kr_1 Bu(t-t)$, **(4)**

where $u(t) = (u_1(t),..., u_n(t))^T$, $r_1 = f'(x^*)$ and $B = \begin{pmatrix} a & b & ... & b \\ b & a & ... & b \\ b & b & ... & a \end{pmatrix}$.

The characteristic equation of (4) is:

$$(l + kr_1(a + (n-1)b)e^{-lt})(l + kr_1(a-b)e^{-lt})^{n-1} = 0 \qquad \textbf{(5)}$$

The analysis of the characteristic equation (5) follows.

Theorem 1

i) when $b > 0, a > 0, a > b, kr_1 > 0$ and $t = 0$, the equilibrium point of (1), be $X^* = (x^*,..., x^*)^T$ is asymptotically stable.

ii) when $b > 0, a > 0, r_1 > 0$, the characteristic equation (5) has pure imaginary roots $l = \pm iw, w > 0$, if and only if :

$$k = k_p = \frac{(2p+1)p}{2t r_1(a + (n-1)b)}, p = 0,2,4,... \quad \textbf{(6)}$$

iii) when the gain parameter k passes through the value k_0, there is a Hopf bifurcation of system (1) at its equilibrium x^*.

The direction and stability of bifurcating periodic solutions

In this section, we study the direction, stability and the period of the bifurcating periodic solutions in system (1). The method we use is based on the normal form theory and the center manifold theorem introduced in [2]. For notational convenience, let $k = k_0^* + m$. Then $m = 0$ is the Hopf bifurcation value for (1). In this section, we assume that the function $f \in C^3(R)$. Expanding the vector field in equations (1) into first, second, third order around x^*, and defining $u(t) = x(t) + x^*$, then equations (1) can be rewritten as:

$$\dot{u}(t) = -k_0 r_1 BF_1(u(t-t)) + F(u(t-t)) + O(|u(t-t)|^4), \qquad \textbf{(7)}$$

where

$$\begin{aligned} F(u(t-t)) &= -\frac{1}{2}k_0 r_2 BF_2(u(t-t)) - \frac{1}{6}k_0 r_3 BF_3(u(t-t)), \\ F_1(u(t-t)) &= (u_1(t-t),..., u_n(t-t))^T, \\ F_2(u(t-t)) &= (u_1(t-t)^2,..., u_n(t-t)^2)^T, \qquad \textbf{(8)} \\ F_3(u(t-t)) &= (u_1(t-t)^3,..., u_n(t-t)^3)^T, \end{aligned}$$

$$r_1 = f'(x^*), r_2 = f''(x^*), r_3 = f'''(x^*), B = \begin{pmatrix} a & b & ... & b \\ b & a & ... & b \\ b & b & ... & a \end{pmatrix}.$$

For $j \in C([-t, 0], C^n)$ and $j^* \in C([-t, 0], C^{n*})$ we define a bilinear form by:

$$(j^*, j) = \overline{j}^*(0)j(0) - \int_{q=-t}^{q}\int_{s=0}^{q} \overline{j}^*(s-q)Bd(q+t)j(s)ds \qquad \textbf{(9)}$$

where d is the Dirac delta function.

In order to determine the Poincare normal form of the model (1), we need to calculate the eigenvector $f(q)$ of the operator A associated and the eigenvector $f^*(q)$ of the adjunct operator A^* associated for equations (7). We can easily verify that for

$w_0 = w(0) = \frac{p}{2t}$,

$$f(q) = e^{iw_0 q} h, \overline{f}^*(q) = e^{-iw_0 q} h, q \in [-t, 0], h = (1,1,...,1)^T \qquad \textbf{(10)}$$

are the eigenvectors of A associated with $\pm iw_0$ and

$$y(s) = \frac{2}{n(1+pi)} e^{iw_0 s} h, \overline{y}(s) = \frac{2}{n(1+pi)} e^{-iw_0 s} h, s \in [0, t], \qquad \textbf{(11)}$$

are the eigenvectors of A^* associated with $\pm iw_0$.

In what follows, we will continue the ideas and use the notations in [4].

Let $u(t-t) = z(t)e^{-iw_0 t} h + \overline{z(t)} e^{iw_0 t} h + \frac{1}{2} w_{20}(-t) z^2(t) + w_{11}(-t) z(t)\overline{z}(t) + \frac{1}{2} w_{02}(-t) z^2(t)$,

where $w_{20}(-t) \in C^n, w_{11}(-t) \in R^n, w_{02}(-t) = \overline{w}_{20}(-t) \in C^n$ and $z(t) = x(t) + iy(t)$, $x(t), y(t) \in R.$ Replacing in (8), we have:

$$F(u(t-t)) = \frac{1}{2} F_{20} z(t)^2 + F_{11} z(t)\overline{z}(t) + \frac{1}{2} F_{02} \overline{z}(t)^2 + \frac{1}{2} F_{21} z(t)^2 \overline{z}(t), \qquad \textbf{(12)}$$

where

$$F_{20} = k_0 r_2 (a + (n-1)b)h, F_{11} = -k_0 r_2 (a + (n-1)b)h, F_{02} = k_0 r_2 (a + (n-1)b)h, \qquad \textbf{(13)}$$
$$F_{21} = -k_0 r_2 iB(w_{20}(-t) - 2w_{11}(-t) + k_0 r_3 (a + (n-1)b))h.$$

Theorem 2

i) The local center manifold $W_{loc}^c(0)$ of system (7), in its point of origin, contains the elements $\tilde{j} \in C^1([-t, 0], R^n)$ given by:

$$\tilde{j}(q) = zf(q) + \overline{zf}(q) + \frac{1}{2} z^2 w_{20}(q) + z\overline{z} w_{11}(q) + \frac{1}{2} \overline{z}^2 w_{02}(q), q \in [-t, 0] \qquad \textbf{(14)}$$

where $z = x_1 + iy_1, (x_1, y_1) \in V_1 \subset R^2$, and

$$w_{20}(q) = -\frac{g_{20}}{iw_0} e^{iw_0 q} h - \frac{\overline{g}_{02}}{3iw_0} e^{-iw_0 q} h + 2e^{iw_0 q} E_1,$$
$$w_{11}(q) = -\frac{g_{11}}{iw_0} e^{iw_0 q} h - \frac{\overline{g}_{11}}{iw_0} e^{-iw_0 q} h + E_2, \qquad \textbf{(15)}$$

and

$$g_{20} = -g_{11} = g_{02} = \frac{2k_0 r_2}{1-pi}(a + (n-1)b),$$
$$g_{21} = -\frac{2k_0 (a + (n-1)b)}{n(1-pi)} (r_2 ih^T (w_{20}(-t) - 2w_{11}(-t)) - r_3 n), \qquad \textbf{(16)}$$
$$E_1 = k_0 r_2 (a + (n-1)B)(k_0 r_1 B - 2iwI)^{-1} h,$$
$$E_2 = -k_0 r_2 (a + (n-1)B)(-k_0 r_1 B)^{-1} h.$$

ii) The solution of equation (1) around x^*, is given by

$$x_i(t) = 2x(t) + r_{20}(x(t)^2 - y(t)^2) + r_{11}(x(t)^2 + y(t)^2) - 2i_{20} x(t) y(t) + x^*, i = 1,...,n \qquad \textbf{(17)}$$

where $(x(t) y(t))$ is the solution of the systems:

$$\begin{aligned}\dot{x}(t) &= -w_0 y(t) + \frac{1}{2}(R_{20} + 2R_{11} + R_{02})x(t)^2 - \frac{1}{2}(R_{20} - 2R_{11} + R_{02})y(t)^2 \\ &+ (I_{02} - I_{20})x(t)y(t) + \frac{1}{2}R_{21}x(t)(x(t)^2 + y(t)^2) - \frac{1}{2}I_{21}y(t)(x(t)^2 + y(t)^2) \\ \dot{y}(t) &= -w_0 x(t) + \frac{1}{2}(I_{20} + 2I_{11} + I_{02})x(t)^2 - \frac{1}{2}(I_{20} - 2I_{11} + I_{02})x(t)^2 \\ &+ (R_{20} - R_{02})x(t)y(t) + \frac{1}{2}R_{21}y(t)(x(t)^2 + y(t)^2) - \frac{1}{2}I_{21}x(t)(x(t)^2 + y(t)^2)\end{aligned} \quad \textbf{(18)}$$

with initial condition:
$x(0) = \mathrm{Re}(y, j), y(0) = \mathrm{Im}(y, j)$ and
$R_{ij} = \mathrm{Re}(g_{ij}), I_{ij} = \mathrm{Im}(g_{ij}), i, j = 0,1,2,\ r_{20} = \mathrm{Re}(v_{20}), r_{11} = \mathrm{Re}(v_{11})$ and

$$\begin{aligned}v_{20} &= -\frac{g_{20}}{iw_0} - \frac{\overline{g}_{02}}{3iw_0} + \frac{r_2(a+(n-1)b)(k_0 r_1(a-b) - 2iw_0)}{(n-1)k_0 r_1^2 - (k_0 r_1 a - 2iw_0)(k_0 r_1(a+(n-2)b) - 2iw_0)} \\ v_{11} &= -\frac{g_{11} + \overline{g}_{11}}{iw_0} + \frac{r_2(a+(n-1)b)(b-a)}{r_1((n-1)b^2 - a(a+(n-2)b))}\end{aligned} \quad \textbf{(19)}$$

Therefore we have the formulas to compute the following parameters:

$$m_2 = -\frac{\mathrm{Re}\,C(0)}{\mathrm{Re}\,M(0)}, T_2 = -\frac{\mathrm{Im}\,C(0) + m_2 \,\mathrm{Im}\,M(0)}{w_0}, b_2 = 2\,\mathrm{Re}\,C(0) \quad \textbf{(20)}$$

where

$$\begin{aligned}C(0) &= \frac{i}{2w}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \\ M(0) &= \left.\frac{dl(t)}{dt}\right|_{l=iw} = \frac{w_0 k_0 r_1(a+(n-1)b)}{1-pi}\end{aligned} \quad \textbf{(21)}$$

Now we can state the main results of this section.

Theorem 3

In the parameter formulas (20), m_2 determines the direction of the Hopf bifurcation: if $m_2 > 0(< 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $k > k_0^*(< k_0^*)$; b_2 determines the stability of the bifurcating periodic solution: the solutions are orbitally stable (unstable) if $b_2 < 0(> 0)$; and T_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_2 > 0(< 0)$.

A numerical example: proportionally –fair congestion controller with REM

Random early marking (REM) marks a packet with probability $(1 - e^{-mw})$ if it finds a workload w already present in the virtual queue, where m is a positive constant. Using a reflected Brownian motion approximation [1], this can be viewed as a mechanism with the following marking function:

$$p(x) = \frac{ms^2 x}{ms^2 x + 2(c - x)}.$$

Here, s^2 denotes the variability of the traffic at the packet level and C is the capacity of the virtual queue. We assume $ms^2 = 0.5$ and let the capacity of the virtual queue be 1Mbps and the round trip delay be 40 ms. Let one round trip time be the unit of time. If the packets are 1000 bytes each, then the virtual queue capacity can be equivalently expressed as $c = 5$ packets per time unit. Let the increase parameter $w = 1$ and $f(x) = xp(x)$ where $p(x) = \frac{x}{20 - 3x}$. The equilibrium rate can be found by solving $x^* p(x^*) = 1$. If $a = 1, b = 0.5, n = 10$ yielding $x^* = 1.65360$.

With Theorem 1, we can determine that $k_0 = 1.114862$, $w_0 = 1.57079$. It following from (22) $m_2 = 0.5831$ (supercritical), $T_2 = -0.8750$ (decreasing), $b_2 = -1.4389$ (stable). These calculations prove that the system equilibrium x^* is stable when $k < k_0$, the critical value $k_0 = 1.114862$, x^* loses its stability and a Hopf bifurcation occurs, i.e, a family of periodic solutions bifurcate out from x^*.

With the help of a program in Maple 8, the following values were obtained for $x^* = 1.65360$ as well as the following graphics: Fig.1 represents the orbit $(t, x(t))$, Fig.2 represents the Poincare application $(x(t), x(t-t))$.

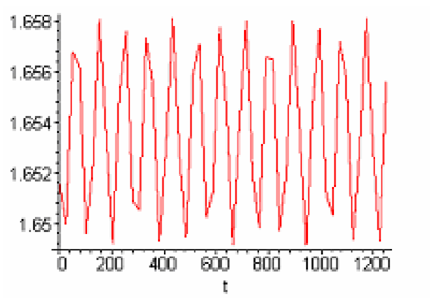

Fig. 1. The orbit $(t, x(t))$

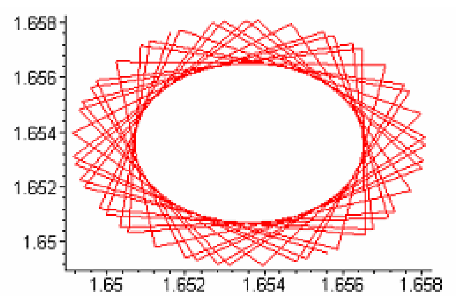

Fig. 2. The Poincare application $(x(t), x(t-t))$

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> #Program in Maple 8 for Internet congestion control model.
> a:=1.;b:=0.5;n:=10;tau:=1.;w:=1.;
> f1(x):=x*(0.06*x -0.2*x^2+0.05*x^3):f2(x):=x^2/(20-3*x):
> eq:=(a+(n-1)*b)*f2(x) -w;sols := [solve(eq,x)];x0:=sols[2];
>D1:=diff(f2(x),x):D2:=diff(f2(x),x,x):D3:=diff(f2(x),x,x,x):ro1:=eval(D1,[x=x0]);ro2:=eval(D2
,[x=x0]);ro3:=eval(D3,[x=x0]);k0:=evalf(Pi)/(2*tau*ro1*(a+(n-1)*b));omega:=evalf(Pi)/(2*tau);
> g20:=2*k0*ro2*(a+(n-1)*b)/(1-evalf(Pi)*I):g11:=-2*k0*ro2*(a+(n-1)*b)/(1-
evalf(Pi)*I):g02:=2*k0*ro2*(a+(n-1)*b)/(1-evalf(Pi)*I): g21:=-2*k0*(a+(n-1)*b)/(1-
evalf(Pi)*I)*(ro2*I*((3*g20-conjugate(g02)+3*(g11+conjugate(g11)))/(3*omega)+ro2*(a+(n-
1)*b)*(b-a)/(ro1((n-1)*b^2-a*(a+(n-2)*b)))-ro2*(a+(n-1)*b)*(k0*ro1*(a-b)-2*I*omega)/((n-
1)*k0*ro1^2*b^2-(k0*ro1*a -2*omega*I)*(k0*ro1*(a+(n-2)*b)-2*omega*I)))-ro3):
> v20:=-g20/(omega*I)-conjugate(g02)/(3*omega*I)+ro2*(a+(n-1)*b)*(k0*ro1*(a-b)-2*omega*I)/((n-
1)*k0*ro1^2*b^2-(k0*ro1*a -2*omega*I)*(k0*ro1*(a+(n-2)*b)-2*omega*I)):v11:=-
(g11+conjugate(g11))/(omega*I)+ro2*(a+(n-1)*b)*(b-a)/(ro1*((n-1)*b^2-a*(a+(n-
2)*b))):w20:=(3*g20-conjugate(g02))/(3*omega)-(ro2*(a+(n-1)*b)*(k0*ro1*(a-b)-2*omega*I)/((n-
1)*k0*ro1^2*b^2-(k0*ro1*a -2*omega*I)*(k0*ro1*(a+(n-2)*b)-2*omega*I))):w11:=-
(g11+conjugate(g11))/omega+ro2*(a+(n-1)*b)*(b-a)/(ro1*((n-1)*b^2-a*(a+(n-2)*b))):
> M:=omega*k0*ro1*(a+(n-1)*b)/(1-tau*I): C1:=(g20*g11 -2*abs(g11)^2-
abs(g02)^2/3)*I/(2*omega)+g21/2: mu2:=-Re(C1)/Re(M);T2:=-
(Im(C1)+mu2*Im(M))/omega;beta2:=2*Re(C1);
>r20:=Re(v20):i20:=Im(v20):r11:=Re(v11):i11:=Im(v11):r220:=Re(w20):i220:=Im(w20):r211:=Re(w11)
:i211:=Im(w11):R20:=Re(g20):R11:=Re(g11):R02:=Re(g02):I20:=Im(g20):I11:=Im(g11):I02:=Im(g02):R
21:=Re(g21):I21:=Im(g21):
> F1(x(t),y(t)):=-omega*y(t)+(R20/2+R11+R02/2)*x(t)^2-(I20-I02)*x(t)*y(t)-(R20/2-
R11+R02/2)*y(t)^2+R21*x(t)*(x(t)^2+y(t)^2)/2-I21*y(t)*(x(t)^2+y(t)^2)/2:
F2(x(t),y(t)):=omega*x(t)+(I20/2+I11+I02/2)*x(t)^2-(I20/2-
I11+I02/2)*y(t)^2+(R20R02)*x(t)*y(t)+R21*y(t)*(x(t)^2+y(t)^2)/2+I21*x(t)*(x(t)^2+y(t)^2)/2:
F3(x(t),y(t)):=2*x(t)+r20*(x(t)^2-y(t)^2)-2*i20*x(t)*y(t)+r11*(x(t)^2+y(t)^2)+x0:
F4(x(t),y(t)):=2*x(t)*cos(omega*tau)+2*y(t)*sin(omega*tau)+r220*(x(t)^2-y(t)^2)-
2*i220*x(t)*y(t)+r211*(x(t)^2+y(t)^2)+x0:
with(DEtools):
> dsys := {diff(x(t),t)=F1(x(t),y(t)),diff(y(t),t)=F2(x(t),y(t)),x(0)=-0.001,y(0)=0.002}:
> dsol :=  dsolve(dsys,numeric):
> plots[odeplot](dsol,[
[t,F3(x(t),y(t))  ,color=red]],0..250*tau,title="Fig.1. The orbit(t,x(t))  ");
plots[odeplot](dsol,[   [F3(x(t),y(t)),F4(x(t),y(t))  ,color=red]],0..250*tau,title="Fig.2.The
Poincare Application  (x(t),x(t-tau))  ");
> OSCEqns := [ diff(x(t),t) =-omega*y(t)+(R20/2+R11+R02/2)*x(t)^2-(I20-I02)*x(t)*y(t)-(R20/2-
R11+R02/2)*y(t)^2+R21*x(t)*(x(t)^2+y(t)^2)/2-I21*y(t)*(x(t)^2+y(t)^2)/2, diff(y(t),t)=
omega*x(t)+(I20/2+I11+I02/2)*x(t)^2-(I20/2-I11+I02/2)*y(t)^2+(R20-
R02)*x(t)*y(t)+R21*y(t)*(x(t)^2+y(t)^2)/2+I21*x(t)*(x(t)^2+y(t)^2)/2 ]:
DEplot3d(OSCEqns, {x(t),y(t)}, t=0..40*tau,[[ x(0)=-0.05, y(0)=0.05]],
x=-0.1..0.1,y=-0.1..0.1,scene=[x(t),y(t),t],stepsize=0.05, linecolour=t,title="Fig.3.The dy-
namics of the system on Wc(0)XR");
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Conclusions

In this paper, an Internet congestion control system model with $n(n \geq 3)$ links and single source has been studied. By using the positive gain parameter as a bifurcation parameter, we have shown that a Hopf bifurcation occurs in such an Internet congestion control model, yielding a family of periodic orbits bifurcating out from the network equilibrium. Simulation results have verified and demonstrated the correctness of the theoretical results.

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