Computational methods used to detect the dynamics in a discrete determinist nonlinear exchange rate model

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In this work we present computational methods used in the study of the exchange rate evolution in a discrete determinist nonlinear model. Our methods also concern the speed for to obtain the results. These methods present how we can obtain with the computer, in a very short time, a very large number of observations, being, at the same time, conclusive on the results obtained.
Keywords: *model, method, forecasts, discount.*

1 Introduction

In this work we present some computational methods used to detect the dynamics as the following model:

(1) $S_t = S_{t-1}^{(2+a)bm_t+(1-a)b} S_{t-3}^{-2bm_t}$

The system (1) is obtained from the following basic equation used to model the exchange rate (see De Grauwe et al. (1993), De Grauwe (1996)):

(2) $S_t = X_t E_t (S_{t+1})^b$

where S_t is the exchange rate at the time t; X_t can be thought as a reduce form describing the structure of the model and the exogenous variables that drives the exchange rate in the period t; E_t (S_{t+1}) is the expectation held today (time t) in the market about next period ($t+1$) exchange rate; b is the discount factor that speculators use to discount the future expected exchange rate ($0 < b < 1$).

The model (2) permits to take into account two components for forecasting, a forecast made by the chartists and a forecast made by the fundamentalists:

(3) $$E_t (S_{t+1})/S_{t-1} = (E_{ct}(S_{t+1})/S_{t-1})^{m_t} (E_{ft}(S_{t+1})/_{t-1})^{1-m_t}$$

where $E_{ct}(S_{t+1})$ and $E_{ft}(S_{t+1})$ are respectively the forecasts made by the chartists and the fundamentalists; m_t is the weight given by the chartists and $1-m_t$ is the weight given by the fundamentalists at the time t.

In this model, for the chartists' behavior one assumed the following model:

(4) $$\frac{E_{ct}(S_{t+1})}{S_{t-1}} = \left(\frac{S_{t-1}}{S_{t-3}}\right)^2$$

The fundamentalists are assumed to calculate the equilibrium exchange rate S_t^*. The steady state can be calculated by imposing $E_{ft} = S_t = S_{t-1}$. This implies that $S_t^* = (X_t)^{1/(1-b)}$. When the current exchange rate is above/below relative to the equilibrium rate, the fundamentalists expect that the future exchange rate to go down/ increase. Their forecasts are obtained from the following model:

(5) $$\frac{E_{ft}(S_{t+1})}{S_{t-1}} = \left(\frac{S_{t-1}^*}{S_{t-1}}\right)^a, \quad a > 0.$$

The weight m_t, given by the chartists, is:

(4) $$m_t = \frac{1}{1 + b(S_{t-1} - S_{t-1}^*)^2}, b > 0.$$

We consider that $X_t = ct = 1$. Now, substituting the equations (4) and (5) in (3), and (3) in (2), we get the system (1).

2. Computational methods used to detect the dynamics of the system (1)

2.1. Mathematical observations

If we denote $s_t = \ln S_t$, then the system (1) can be rewritten in the following way:

(5) $$s_t=\left(\frac{(2+a)b}{1+b(e^{s_{t-1}}-1)^2}+(1-a)b\right)s_{t-1}-\frac{2b}{1+b(e^{s_{t-1}}-1)^2}s_{t-3}$$

with $s_t \in R, \forall t \in Z$.

We can also rewrite the system (5) in a vectorial form and we get the expression:

(6) $$(s_{t+1},s_{t+2},s_{t+3})=F(s_t,s_{t+1},s_{t+2})$$

if we define the function $F:R^3\to R^3$, such that $F(x,y,z)=(F_1(x,y,z),F_2(x,y,z),F_3(x,y,z))$, where $F_1(x,y,z)=y$, $F_2(x,y,z)=z$ and $F_3(x,y,z)=\left(\frac{(2+a)b}{1+b(e^z-1)^2}+(1-a)b\right)z-\frac{2b}{1+b(e^z-1)^2}x$.

We use the vectorial form to detect the fixed point, the periodic cycles and the Lyapunov exponents.

2.2. The fixed point of the system (6)

Proposition 1. In the case in which $b\in(0,1)$, $a>0$, $b>0$, the system (6) has the fixed point (0,0,0).

The Jacobian matrix of F at (0,0,0) is the matrix

(7) $$J_{(0,0,0)}=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2b & 0 & 3b\end{pmatrix}$$

The eigenvalues are the solutions of the following equation:

(8) $$l^3-3bl^2+2b=0$$

Using *Mathematica*, we obtain that for $b\in(0,0.3165)$ the eigenvalues verify $|l_i|<1, i=\overline{1,3}$ and for $b\in(0.3165,1)$ the eigenvalues verify $|l_{1,2}|>1$ and $|l_3|<1$. This means that for $b\in(0,0.3165)$ the fixed point (0,0,0) is stable and for $b\in(0.3165,1)$ this point is unstable.

The functions used in *Mathematica* are the following:

```
m = {{0,1,0},{0,0,1},{-2*b,0,3*b}}
Abs[Eigenvalues[m]]
```
for $b\in(0,1)$.

We make the observation that using a computational method we give, in this case, a qualitative result, not only a quantitative result. The mathematical method is more complicate and the numerical results close the problem.

For *b=0.3165* we observe a bifurcation. After bifurcation, the fixed point is unstable and it is surrounded by a limit cycle that is stable (we observe this from simulations). Within a neighbourhood of the fixed point, all the orbits starting outside or inside the closed invariant curve, except at the origin, tend to the limit cycle under iterations of F. This is a Neimark-Sacker bifurcation.

2.3. Attracting p-cycle, quasiperiodic attractor, and strange attractor.

We recall that a set *A* is an attracting set with the fundamental neighbourhood *U*, if it verifies the following properties (see Ruelle (1989)):

1) attractivity: for every open set $V\supset A$, $F^t(U)\subset V$ for all sufficiently large t.

2) invariance: $F^t(A)=A$, for all *t*.

3) *A* is minimal: there is no proper subset of *A* that satisfies conditions 1 and 2.

The basin of attraction is the set of initial points *x* such that $F^t(x)$ is close to *A* as $t\to\infty$.

It is possible to classify the different attractors: attracting fixed point, attracting n-cycle, quasiperiodic attractor and strange attractor. An attractor, as an experimental object, gives a global description of the asymptotic behavior of a dynamical system.

Now, we recall that a point $(x,y,z)\in R^3$ is a *period-p point* of the system (6) if $(x,y,z)=F^p(x,y,z)$ and $(x,y,z)\neq F^k(x,y,z)$ for $k=\overline{1,p-1}$.

Chaos – is defined to be aperiodic, bounded dynamics in a deterministic system with sensitive dependence on the initial conditions.
Aperiodic means that the same state is never repeated twice. Bounded means that on successive iterations the state stays in a finite range and does not approach $\pm\infty$. Sensitive dependence on initial conditions means that two points that are initially close will drift apart as time proceeds.
Quasiperiodcity – is another type of aperiodic behavior in which two points that are initially close will remain close over time.
Now, we present computational methods used to detect the attractors. We present the implementation in *Excel*, using functions and macros.
In the cells B1, B2, B3 we give the values for the parameters a, b and *b*, respectively. In the cells B6, B7, B8 we introduce the values for the initial conditions. In the cells B9 we introduce the formula:
B9 =((2+$B$1) * $B$3 / (1 + $B$2 * (EXP(B8) - 1)^2) + (1-$B$1) * $B$3) * B8 –2 * $B$3 / (1+$B$2 * (EXP(B8)-1)^2) * B6.
This formula is copied from the cells B9 in the domain B10:B20005. We make a chart in the space (s_t, s_{t+1}) for to observe easily the dynamics. Generally, 20000 points are not sufficient to detect the behavior type; we have need to many values. The using of many cells with formulas is not recommended, because it induces a slow speed for to obtain the results. For to observe many values on the trajectory in a short time, we propose the following algorithm:
We observe the first 20000 points and for to observe the following (20000) points we make the last three points as the initial conditions and we repeat this. In the chart we can observe, after some repetitions, the type of dynamics (see, for example, the Figure 1).
It is recommended to use macros, for to obtain the results in a short time. Now, we present the macros used for to change the initial conditions:

```
Sub schimbare()
  For i = 1 To 10
   Application.Goto Reference:="R20005C1"
   Range("B20003:C20005").Select
   Selection.Copy
   Application.Goto Reference:="R1C1"
   Range("B6").Select
   Selection.PasteSpecial Paste:=xlValues, Operation:=xlNone, SkipBlanks:= _
      False, Transpose:=False
   Range("E4").Select
  Next i
End Sub
```

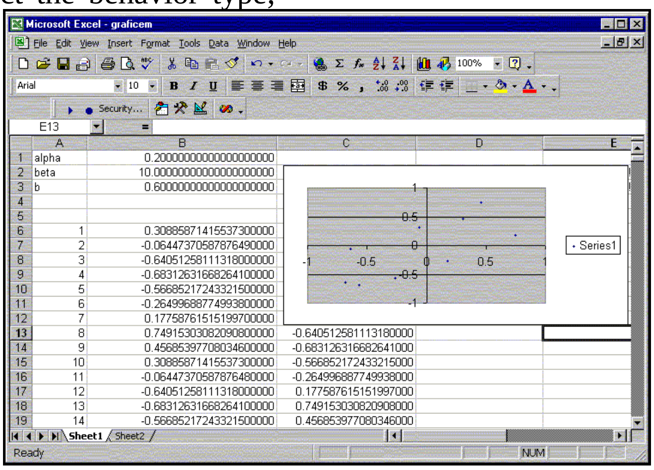

Fig. 1. Implementation of exchange rate evolution

In this case we observe the dynamics after the 200 000 iterations. If the results are not yet conclusive we can repeat the algorithm.
When we are in the case of a bifurcation (for example between periodicity and chaos) is very difficult to observe from

chart the dynamics. In this case it is necessary to repeat the algorithm for many iterations, for to detect the type of behavior.
In the case in which the attractor seems to be an attracting p-cycle, for to know the period, we use the formulas:
=COUNTIF(B6:B20005,B6)
=20000/C4

The results are not exactly, there are only approximations and for to be sure on the results, we calculate the Lyapunov exponents. If we find that the Lyapunov exponents are negative, we can conclude that there is a periodic cycle.
The Lyapunov exponents l_1, l_2, l_3 are given by

$$\{e^{l_1}, e^{l_2}, e^{l_3}\} = \lim_{n\to\infty}\left\{ eigenvalues \quad of \quad \left(\prod_{t=0}^{n-1} J(F(s_t, s_{t+1}, s_{t+2}))\right)^{\frac{1}{n}} \right\}$$

where $J(F(s_t, s_{t+1}, s_{t+2}))$ represents the Jacobian matrix of the function F. For a period-p point the Lyapunov exponents are

$$\{e^{l_1}, e^{l_2}, e^{l_3}\} = \left\{ eigenvalues \quad of \quad \left(\prod_{t=0}^{p-1} J(F(s_t, s_{t+1}, s_{t+2}))\right)^{\frac{1}{p}} \right\}$$

When the system (6) displays a chaotic behavior, we use the method proposed in Eckmann and Ruelle (1985) based on the Householder QR factorization, to compute the Lyapunov exponents.
We recall now that for an attracting period-n cycle the Lyapunov exponents are negative; in case of a bifurcation point, at least a Lyapunov exponent is zero; for a limit cycle one
Lyapunov exponent is zero and the others are negative and for a chaotic behavior the highest Lyapunov exponent is positive and the sum of the Lyapunov exponents is negative.
The method proposed in Eckmann and Ruelle (1985) can be used (in implementation methods) for each attractor type. An implementation method for this algorithm is presented by Balint and Voicu, 2001. Using this method we have need to observe many approximations of the Lyapunov exponents.
The definition of the Lyapunov exponents can be used in implementation methods only for the cases in which the trajectory is a *n-cycle*. In this case we have need to observe only n iterations and we get the exact values of the Lyapunov exponents.
For example, if we choose $a = 0.2$, $b = 10$ and $b=0.6$, and the initial condition is $(s_1, s_2, s_3) = (0.02, -0.02, 0.07)$, we find that the trajectory tends to a period-9 cycle. In the Figure 2 we represent the first 4000 points in the space (s_t, s_{t+1}).

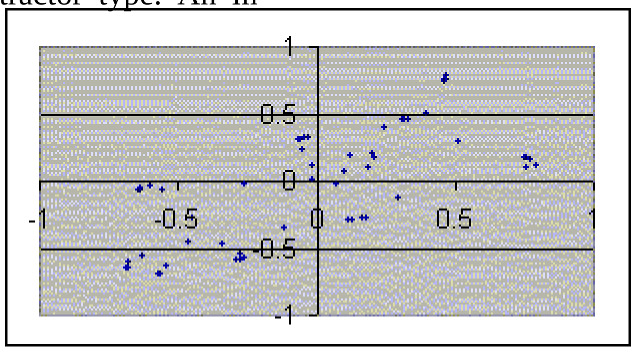

Fig. 2. The firsts 4000 points of a trajectory, which tends to a period-9 cycle

In the Figure 1 we have the image of the trajectory for 4000 points, where the firsts 200 000 are ignored. We also observe from this figure that the values are given with a 10^{-17} precision and, in addition, these are only approximation. For this reason we have need to calculate the Lyapunov exponents. We obtain that l_1=-0.1253, l_2=-0.1254 and l_3=-0.5853, which confirm that there is a periodic cycle.

In the case in which $a = 0.2$, $b = 10$ and *b=0.4*, and the initial condition is $(s_1, s_2, s_3) = (0.02, -0.02, 0.07)$ we find that the trajectory tends to a limit cycle.

In the Figure 3b we represent the trajectory in the space (s_t, s_{t+1}) for the firsts 4000 points. We observe that the trajectory seems to be a limit cycle. In the Figure 3a we observe that there it is not a sensitive dependence on the initial conditions (the values are represented in the space (t, s_t) for initial conditions $(s_1, s_2, s_3) = (0.02, -0.02, 0.07)$ and $(s_1, s_2, s_3) = (0.02, -0.021, 0.07)$ for the firsts 50 points). In the Figure 3c we represent the image of the quasiperioidc attractor for 4000 iterations, where the firsts 200 000 are ignored.

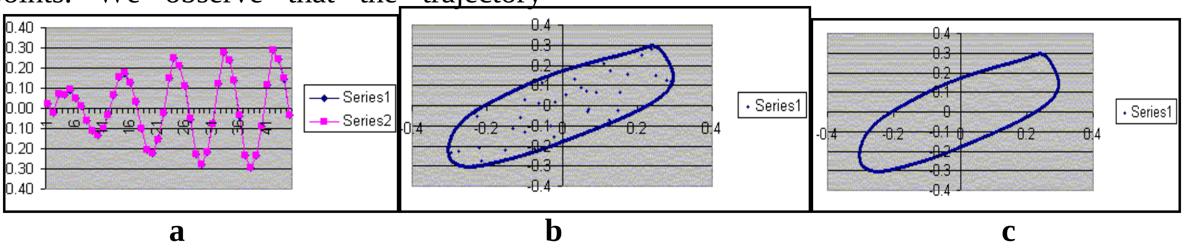

Fig. 3. Limit cycle

We obtain that the Lyapunov exponents are l_1=0, l_2=-0.14875 and l_3=-0.41462, which confirm that there it is a limit cycle.

In the case in which $a = 2$, $b = 600$ and *b=0.6* and the initial condition is $(s_1, s_2, s_3) = (0.02, -0.02, 0.07)$ we find that the trajectory tends to a strange attractor.

In the Figure 4b / Figure 4c we represent the trajectory in the space (s_t, s_{t+1}) / (s_t, s_{t+2}) for the firsts 4000 points. In the Figure 4a we observe that there it is a sensitive dependence on the initial conditions (the values are represented in the space (t, s_t) for initial conditions for the firsts 100 points: $(s_1, s_2, s_3) = (0.02, -0.02, 0.07)$ and $(s_1, s_2, s_3) = (0.02, -0.021, 0.07)$.

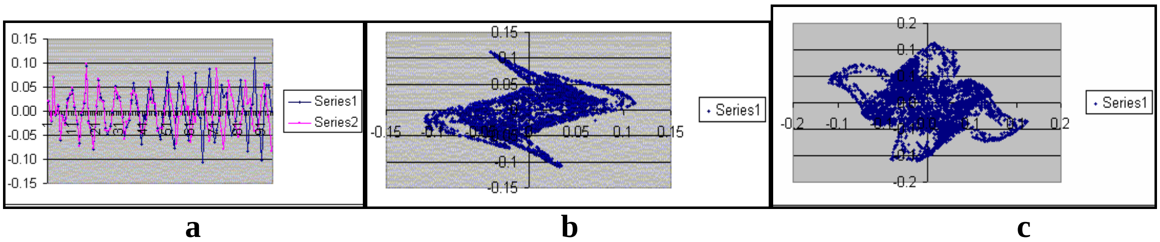

Fig. 4. Chaotic behavior

We obtain that the Lyapunov exponents are l_1=0.2337, l_2=-0.2471 and l_3=-0.4288, which confirm that there it exists a chaotic behavior.

For to obtain the image of the strange attractor in the space (s_t, s_{t+1}, s_{t+2}) we use in *Mathematica* the following algorithm (we obtain the figure 5):

```
s[t_] :=
 s[t] = ((2 + α) * b / (1 + β * (Exp[s[t - 1]] - 1) ^ 2) + (1 - α) * b) * s[t - 1] -
   2 * b / (1 + β * (Exp[s[t - 1]] - 1) ^ 2) * s[t - 3]
α = 2 ; β = 600; b = 0.6; s[0] = 0.02; s[1] = -0.02 ; s[2] = 0.07;
l = Table[N[{s[t], s[t + 1], s[t + 2]}], {t, 1500, 11500}]
ScatterPlot3D[l]
```

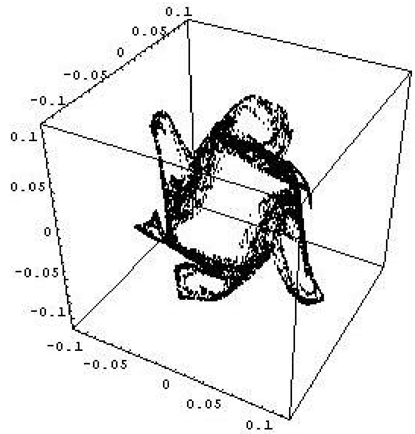

Fig. 5. Strange attractor in the space (s_t, s_{t+1}, s_{t+2})

3. Conclusion

For the system (1) mathematical and empirical study and an economical interpretation of the results are presented by Guégan D., Voicu M.C (2000), Voicu M. et all.(2001). In this work we have presented the computational methods used to detect the exchange rate evolution. We have implemented the algorithms in *Excel*, because using macros we can obtain the result in a very short time. In the empirical study this aspect is very important. We have use *Mthematica* only in the cases in which we cannot use the instruments from *Excel*. With the methods presented we can detect (empirical) only attractors. The using of the Lyapunov exponents confirms the nature of attractor. Given the nonlinear nature of the system, we cannot obtain many mathematical results. For this reason, it is necessary an empirical study for to know the dynamics. Fixing the values of the parameters and initial conditions and using these implementation methods, we can detect the exchange rate evolution.

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