The supply of a public good in a single stage or a multiple stage game having incomplete information

Senior lect. Stelian STANCU, PhD.
Department of Cybernetics, Academy of Economic Studies Bucharest

Goods can be divided in two categories: we will call exclusive a good if it is relatively easy to prohibit a person from using this it, while a nonexclusive implies that it is basically impossible (or very expensive) to prohibit a person from using it.
Furthermore, a good is called nonrival if the use of an additional item of the good implies a zero marginal production cost.
The two concepts – non-exclusivity and non-rivality – are generally linked to each other (an immediate example is the national security system).
A good is public (pure) if it is nonexclusive. The public goods are usually nonrival, too, but this is not a necessary condition .Private goods are different from the public ones through the two qualities.
Keywords: *exclusive goods, public goods, incomplete, information, game, player, equilibrium*

1 The supply of a public good in a single stage game

We will now consider two players i=1,2. Each player gains utility from the supply of the public good, but each of them would like the other one to support the cost of the supply. The players will simultaneously decide whether they will take part in the financing of the good or not.
We will consider that each player gains a utility unit if at least one player pays, or no utility unit if neither of the players contributes to the financing, the cost of the contribution being c_i for the i player.
The benefits from the supply of the public good (a utility unit) stand for common information, but only the player himself knows his costs.
Both players suppose that (common information) they have the same distribution P(•) in the $[\underline{c}, \overline{c}]$ interval, where $\underline{c}<1<\overline{c}$, $P(\underline{c})=0$, $P(\overline{c})=1$, and the distribution function P (·) is strictly ascendant.
A pure strategy is a function $s_i(c_i):[\underline{c}, \overline{c}]\rightarrow\{0,1\}$, where $s_i(c_i)=1$ implies that the i player supplies and $s_i(c_i)=0$ means that the i player doesn't supply.
If both players supply, the utilities they gain are $(u_1,u_2)=(1-c_1,1-c_2)$.
If the j player supplies, the utilities are: $(u_i, u_j)=(1,1-c_j)$.
If neither of the players supplies, the utilities are: $(u_1,u_2)=(0,0)$.
The utility the i player gains is: u_i (s_i, sj, ci)=max (s_1,s2)-$c_i\cdot s_i$
Note: The u_i utility doesn't depend on c_j, $j\neq i$.
The Bayes equilibrium is a pair of strategies $(s_1^*(\cdot),s2^*(\cdot))$, and for every i player and every possible value c_i, the strategy $s_i^*(c_i)$ makes the maximum of

$$E\ u_i\ (s_i,\ sj^*(c_j),\ ci),$$

where E is the waiting operator.
Let $z_j=P\ (s_j^*(c_j)=1)$ be the probability that at the equilibrium, the j player to supply.
In order to maximize the expected utility, the i player will supply if:

$$c_i<1\cdot(1-z_j),$$

which stands for his benefit multiplied with the probability that the j player doesn't supply. In these terms, we have:

$$s_i^*(c_i)=1,\ \text{if}\ c_i<1-z_j$$
$$s_i^*(c_i)=0,\ \text{if}\ c_i>1-z_j.$$

Note: The case when $c_i=1-z_j$ stands for the indifference between supplying and not supplying, but as $P(\cdot)$ is a continuous function, the probability of this particular case is 0.

Then the types of the i player supplying are in the $[\underline{c}, c_i^*]$ interval, and therefore the i player supplies only if only if his costs are low enough (as a rule, if $c_i^*<\underline{c}$, then $[\underline{c}, c_i^*]=\Phi$).

Similarly, the j player supplies only if $c_j\in[\underline{c}, c_j^*]$.

As $z_j=P(\underline{c}\le c_j\le c_j^*)=P(c_j^*)$, the equilibrium levels c_i^* are related this way:

$$c_i^*=1-p(c_j^*)$$

For example, if P is a uniform function in the [0,2] interval and $P(c)=c/2$, then $c_i^*=\frac{2}{3}$.

If a player doesn't supply, then his expected utility is $P(c_i^*)=\frac{1}{3}$, and if a player having the c^* cost contributes, his utility is $1-c_i^*=\frac{1}{3}$.

A player contributes if his cost is in the $(\frac{2}{3},1)$ interval, even if he has a cost lower than his benefit and even if the probability that the other player doesn't supply is $1-P(c_i^*)=\frac{2}{3}$.

If we suppose that $\underline{c}\ge 1-P(1)$, the game has two Nash equilibrium points, which are asymmetric. In these cases, one player never supplies and the other contributes if $c\le 1$.

The equilibrium case in which the first player never contributes is preferred as the minimum cost $\underline{c}$ is higher than the benefit $1\cdot(1-P(1))$.

The player who supplies for $c\le 1$ plays an optimal strategy (if he wouldn't supply, then the probability of getting the good is 0).

2. The process of eliminating the strictly dominating strategies

If we admit that the lowest cost possible $\underline{c}$ is $\underline{c}>1-P(1)$, then the process stops after the first iteration: for all the values in the $[\underline{c},1]$ interval, neither of the strategies implying that the player supplies or not is a dominating one.

We suppose that $\underline{c}<1-P(1)$. In these terms, there is a unique value $c^*=1-P(1-P(c^*))$.

In the first iteration of the process, neither of the two players having a cost higher than 1 doesn't supply (the supplying strategy is a strictly dominating for all $c_i\in(c_1, \bar{c}]$, where $c_1=1$).

In the second iteration, not supplying is a strictly dominating strategy for all $c_i\in[\underline{c}, c_2)$, where $c_2=1-P(1)=1-P(c_1)$.

The optimal strategy for the cases $c_i\in[c_2,c_1]$ depends on the cases $c_j\in[c_2,c_1]$. Furthermore, we must keep in mind that none of the strategies for these cases can be eliminated in the second iteration.

In the third iteration, the cases when the cost is close to 1 should not supply, due to the fact that the cost of the contribution is close to the particular value of the public good and the probability that the second player supplies is at least $P(c_2)$. So, if $c_i>c_3=1-P(c_2)$, then supplying strictly dominating strategy for the player i.

Going further with the iterations of the eliminating process, in the 2k+1 stage (k=0,1,…) we reach the result that supplying is a strictly dominating strategy for the cases higher than $c_{2\cdot k+1}=1-P(c_{2\cdot k})$.

In the 2k (k=0,1,…) iteration, not contributing is a strictly dominating strategy for the cases lower than $c_{2k}=1-P(c_{2\cdot k}-1)$.

The arrays $\{c_{2\cdot k+1}\}_{k=0,1,..}$ and $\{c_{2\cdot k}\}_{k=0,1,..}$ are *strictly descendant* or *strictly ascendant*.

Since they are bounded, they converge to c^+ or c^-.

Since P is a continuous function, $c^+=1-P(c^-)$ and $c^{--}=1-P(c^+)$. If there is a unique value c^* so as $c^*=1-P(1-P(c^*))$, which is the condition for the uniqueness of the Nash equilibrium, then $c^+=c^{--}=c^*$, and the game is solvable through the iteration of the strict dominance (the iterative eliminating of the strictly dominating strategies).

3. The supply of a public good in a game with more stages (example on two stages)

We consider the same game previously analyzed: there are two players i=1,2, but the

game is repeated, in each period t=0,1 the players deciding whether they will contribute to the financing of the good.
During each period, each player gains a utility unit if at least one of them has contributed and, no utility units if neither of them has supplied for the good. The cost of the contribution of the i player is c_i , the same in the two periods.
We will suppose that each player updates his utility, the updating rate being $0<\delta<1$. Then, the objective function for each player is the sum of the utilities in the first stage and in the second stage (which is updated).
Both players anticipate that c_i has the repartition P(·) in the $[0, \bar{c}]$ interval, $\bar{c}>1$ (the cost c_i is private information).
From the previous analysis we have that if $c^*=1-P(1-P(c^*))$ has a unique solution, then the game with a single stage has a Bayes equilibrium, and c* is given by the equation $c^*=1-P(c^*)$, meaning that the cost of the contribution is equal to the probability that the opponent doesn't supply.
In the multiple stage game, the space of the action taken by each player is {0,1}.
A strategy of the i player is $\sigma_i^0(1|c_i)$ (the probability that he contributes during the first period, when his cost is c_i) and $\sigma_i^1(1|h^1,c_1)$, the probability that he supplies during the second period, when his cost is c_i and the past action is $h^1\in\{00,01,10,11\}$.
We will analyze the Bayes equilibrium of the second period, taking into consideration the anticipations worked out on the equilibrium of the first period.

- *neither of the players supplies*

Both players know (in the second stage) that his opponent has a cost higher than $\hat{c}$.
The anticipations are:

$$p(c_i|00)=\frac{p(c_i)-p(\hat{c})}{1-p(\hat{c})}, \text{ for } c_i\in[\hat{c},\bar{c}] \quad (1.)$$

and $p(c_i|00)=0$, for $c_i\leq \hat{c}$.
In the equilibrium situation, (in the second stage), each player i supplies only if $\hat{c}\leq c_i\leq c_0$, the level of his cost c_0 being the same as the probability $\frac{1-p(c^0)}{1-p(\hat{c})}$ that his opponent doesn't supply, a Bayes equilibrium in the second stage implying as a rule this level of the cost.
The player having c_0 supplies during the second stage if no one has supplied in the first stage, his utility in the second stage being $v_{00}(\hat{c})=1-\hat{c}$.

-*both players supply*

Then $p(c_i|11)=\frac{p(c_i)}{p(\hat{c})}$, $c_i\in[0, \hat{c}]$ (2.)
and $P(c_i|11)=1$, $c_i\in[\hat{c}, \bar{c}]$.
In the equilibrium situation of the second stage, each player i contributes only if $c_i\leq\tilde{c}$, where $0<\tilde{c}<\hat{c}$. Each player's cost is equal to the conditioned probability that his opponent doesn't supply:

$$\tilde{c}=\frac{p(\hat{c})-p(\tilde{c})}{p(\hat{c})} \quad (3.)$$

As a particular case, the player with $\hat{c}$ doesn't supply, so the utility he gains during the second stage is:

$$v_{11}(\hat{c})=P(\tilde{c})/P(\hat{c}).$$

- *a single player supplies*

We admit that the i player has supplied in the 0 stage, and the j player hasn't. Then $c_i\leq\hat{c}$ and $c_j\geq\hat{c}$.
The equilibrium of the first stage is that in which the i player has contributed (and $\hat{c}<1$), and the j player hasn't.
The utility units gained in the second stage by the player having $\hat{c}$ are therefore:

$$v_{10}(\hat{c})=1-\hat{c} \text{ and } v_{01}(\hat{c})=1$$

We will analyze the equilibrium situation of the first stage. The player with $\hat{c}$ is indifferent to the strategies of supplying or not, then:

$$1-\hat{c}+\delta\cdot\{P(\hat{c})\cdot v_{11}(\hat{c})+[1-P(\hat{c})]\cdot v_{10}(\hat{c})\} \quad (4.)$$
$$=P(\hat{c})+\delta\cdot\{P(\hat{c})\cdot v_{01}(\hat{c})+[1-P(\hat{c})]\cdot v_{00}(\hat{c})\}$$

Using the formulas for the utility units gained in the second stage and the equation (3.), we find that:

$$1-P(\hat{c})=\hat{c}+\delta\cdot P(\hat{c})\cdot\tilde{c} \quad (5.)$$

From the equations (3.) and (5.) we can define $\hat{c}$.

We can interpret the (5.) equation this way: supplying in the first stage, the player with $\hat{c}$ spends $\hat{c}$, but he can use the public good, otherwise he couldn't. If he doesn't contribute in the first stage, then he will determine his opponent to supply in the second stage, and if he supplies he determines his opponent to be less willing to contribute, thus choosing to contribute in the second stage only if he has a cost lower than the $\tilde{c}$ level.

As the utility of a player in the second stage when he doesn't supply is independent from his cost, and the player having $\tilde{c}$ is indifferent to the strategies of supplying or not, in the case when both of the players have supplied in the first stage, the player with $\hat{c}$ gains 1-(1-$\tilde{c}$)= $\tilde{c}$, when the cost of his opponent is lower than $\hat{c}$.

The (5.) equation implies that $\hat{c}$<c*. In this equilibrium situation, the contribution is lower in the first stage of the two-stage game than in the single-stage game previously analyzed.

References

[1] Laffont J.J., Tirole J., *A Theory of Incentives in Procurement and Regulation,* M.I.T. Press, 1993

[2] Salanie B. *Teorie des contrats*, Ed. Economica, Paris, 1994

[3] Stancu S., Andrei T., *Microeconomie-teorie si aplicatii,* Editura ALL, Bucuresti, 1997:

[4] Stancu S., *Teoria contractelor-negocierea si derularea contractelor economice,* Editura ASE, Bucuresti, 2002;

[5] Tirole J., *Industrial Organization,* M.I.T. Press, 1988