Minimum effort of reorganisation distance

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Abstract: *The directional efficiency measures reveal how inefficient (or efficient) is a decision-making unit on the direction selected but it nothing says about the purport of this direction and who is the direction who must be selected. Also, the directional models fix from the beginning same direction for every production unit albeit they are different situated respecting frontier. It is very probable that each production unit prefer certain directions, these directions depending by her present structure, present dimension, future objectives etc. We propose in this section to introduce the "effort" function that a possible answer of this aspect of analysis.*
Key words: *Efficiency measures, Directional efficiency measures, Efficient frontier, Data envelopment analysis.*

1 Effort function

The reasoning following are based at hypothesis that the decision-making units propose to become efficient.

Because efficiency measurement is done relative to efficient frontier and the projection point of decision-making unit on this direction is the efficient decision-making unit appropriate (so desirable) it is natural to see how difficult is this action and how is the most facile alternative.

In this purpose we introduce the "*effort*" function defined by:

$f_{eff}: R_+^{m+n} \times R_+^{m+n} \rightarrow R_+$

where $f_{eff}(x_0,y_0;x,y)$ = "the minimum necessary effort to pass at (x_0,y_0) structure in (x,y) structure"

If $\|x,y\|$ is norm of (x,y) vector then the value:

$$f_{eff}(x_0,y_0;x_0 + \theta\frac{g_x}{\|g_x,g_y\|}, y_0 + \theta\frac{g_y}{\|g_x,g_y\|}) = f_{eff}(x_0,y_0;(x_0,y_0) + \theta\frac{(g_x,g_y)}{\|g_x,g_y\|})$$

represent the minimum necessary effort for the reorganisation with θ units par (g_x,g_y) direction starting of (x_0,y_0).

We suppose to effort function have the property:

a) f_{eff} is continue

b) $f_{eff} \geq 0$ $(\forall)$ $(x_0,y_0;x,y) \in R_+^{m+n} \times R_+^{m+n}$

c) $f_{eff}(x_0,y_0;x,y) = 0 \Leftrightarrow (x_0,y_0) = (x,y)$

We define the distance function $D_R(x,y)$ by:

$$D_R(x_0,y_0)=\begin{cases}\min\limits_{\substack{g_x\in R_+^m\\ g_y\in R_+^n}} f_{eff}(x_0,y_0;(x_0y_0)+D_T(x_0,y_0;g_x,g_y)\cdot(-g_y,g_y)) & \text{if exist}(x,y)\in T,(x,y)\geq(x_0y_0)\\ -\min\limits_{\substack{g_x\in R_+^m\\ g_y\in R_+^n}} f_{eff}((x_0y_0)+D_T(x_0,y_0;g_x,g_y)\cdot(-g_y,g_y);x_0,y_0) & \text{if it is not}\end{cases}$$

where $D_T(x_0,y_0:g_x,g_y)$ = the directional technology distance function for (x_0,y_0) on (g_x,g_y) direction.

Practically, this distance represent, for a feasible production (x_0,y_0), smallest effort necessary for become efficient and, for an infeasible production (x_0,y_0), smallest effort with which one can reach this performance on the basis of technology gives.

For practical compute of this distance we make following observations.

1. If we note:

$$f_{eff}(x_0,y_0;(x_0y_0)+D_T(x_0,y_0;g_x,g_y)\cdot(-g_y,g_y)) = f_{eff\,(x_0,y_0)}(g_x,g_y)$$

then $f_{eff\,(x_0,y_0)}(g_x,g_y) = f_{eff\,(x_0,y_0)}(\lambda g_x,\lambda g_y)$, $\lambda > 0$.

Indeed, because $D_T(x,y;\ \lambda g_x,\lambda g_y) = \frac{1}{l} D_T(x,y;\ g_x,g_y)$, result:

$$f_{eff\,(x_0,y_0)}(\lambda g_x,\lambda g_y) = f_{eff}(x_0,y_0;(x_0 y_0) + D_T(x_0,y_0;\ l g_x,\ l g_y)\cdot(-l g_y,\ l g_y)) =$$

$$f_{eff}\left(x_0,y_0;(x_0 y_0) + \frac{1}{l} D_T(x_0,y_0;g_x,g_y)\cdot(-l g_y,\ l g_y)\right) =$$

$$f_{eff}(x_0,y_0;(x_0 y_0) + D_T(x_0,y_0;g_x,g_y)\cdot(-g_y,g_y))$$

So, for compute the distance function $D_R(x,y)$ is sufficient to compute only on normat direction.

2. If T is closely and f_{eff} is continue then the optimum problem:

$$\min_{\substack{g_x\in R_+^m \\ g_y\in R_+^n}} f_{eff}(x_0,y_0;(x_0 y_0) + D_T(x_0,y_0;g_x,g_y)\cdot(-g_y,g_y))$$

have solution. Can be much direction for which expects the minimum of the problem, therefore much projection of (x_0,y_0) on efficient frontier but everyone necessitating same effort for be reach.

For a direction $(g_x^{\min}, g_y^{\min})$ which gives the minimum, the projection point(s) of (x_0,y_0) on efficient frontier is:

$$(\alpha,\beta) = (x_0 y_0) + D_T(x_0,y_0;g_x^{\min},g_y^{\min})\cdot(-g_x^{\min},g_y^{\min})$$

3. If T technology is free disposable then the projection point(s) of (x_0,y_0) is more efficient than (x_0,y_0) if $D_R(x_0,y_0) > 0$ and less efficient if $D_R(x_0,y_0) < 0$.

Indeed, if $D_R(x_0,y_0) > 0$ then exist $(x,y) \in T$, $(x,y) \neq (x_0,y_0)$ and $x \le x_0$, $y \ge y_0$. Pursuant to free disposable hypothesis $(x_0,y_0) \in T$. Because $(x_0,y_0) \in T \Leftrightarrow D_T(x,y;\ g_x,g_y) \ge 0$ for each direction (g_x,g_y) with $(g_x,g_y) \ge 0$ and $(g_x,g_y) \neq 0 \Rightarrow (-\alpha,\beta) \ge (-x_0,y_0)$ that is to say (α,β) is more efficient than (x_0,y_0).

if $D_R(x_0,y_0) < 0 \Rightarrow (\nexists)\ (x,y) \in T$ with $(-x,y) \ge (-x_0,y_0) \Rightarrow$

$\Rightarrow D_T(x,y;\ g_x,g_y) < 0$ for each direction (g_x,g_y) with $(g_x,g_y) \ge 0$ and $(g_x,g_y) \neq 0 \Rightarrow$

$$\Rightarrow \begin{cases} D_T(x_0,y_0;g_x^{\min},g_y^{\min})\cdot(-g_x^{\min}) \ge 0 \\ D_T(x_0,y_0;g_x^{\min},g_y^{\min})\cdot g_y^{\min} \le 0 \end{cases} \Rightarrow \begin{cases} a \ge x_0 \\ b \le y_0 \end{cases} \Rightarrow$$

$\Rightarrow (-\alpha,\beta) \le (-x_0,y_0)$ that is to say (α,β) is less efficient than (x_0,y_0).

4. $D_R(x_0,y_0) = 0 \Leftrightarrow (\exists)\ g = (g_x,g_y),\ g \neq 0$ and $g \ge 0$ that $(x_0,y_0) \in Isoq_g(T)$

``$\Rightarrow$``

$D_R(x_0,y_0) = 0 \Rightarrow (\exists)\ (g_x,g_y)$ that $f_{eff}(x_0,y_0;(x_0 y_0) + D_T(x_0,y_0;g_x,g_y)\cdot(-g_y,g_y)) = 0$

$$\Rightarrow (x_0,y_0) = (x_0 y_0) + D_T(x_0,y_0;g_x,g_y)\cdot(-g_y,g_y) \Rightarrow$$

$$\Rightarrow D_T(x_0,y_0;g_x,g_y)\cdot(-g_y,g_y) = 0 \Rightarrow$$

$$\Rightarrow D_T(x_0,y_0;g_x,g_y) = 0 \Rightarrow (x_0,y_0) \in Isoq_g(T)$$

``$\Leftarrow$``

$(x_0,y_0) \in Isoq_g(T) \Rightarrow D_T(x_0,y_0;g_x,g_y) = 0 \Rightarrow (x_0,y_0) \in T \Rightarrow$

$$\Rightarrow f_{eff}(x_0,y_0;(x_0 y_0) + D_T(x_0,y_0;g_x,g_y)\cdot(-g_y,g_y)) = f_{eff}(x_0,y_0;x_0 y_0) = 0 \Rightarrow$$

$$\Rightarrow \min_{\substack{g_x\in R_+^m \\ g_y\in R_+^n}} f_{eff}(x_0,y_0;(x_0 y_0) + D_T(x_0,y_0;g_x,g_y)\cdot(-g_y,g_y)) = 0 \Rightarrow$$

$$\Rightarrow D_R(x_0,y_0) = 0$$

2. Algorithm for compute the efficiency measurement

For a technology given by k observed DMU the practical compute of distance function $D_R(x,y)$ can be realised thus:

step 1. We find, using one among the measures who eliminate the slacks, i.e. Färe-Lovell, Zieschang etc, the DMUs that are efficient in input and in output.

step 2. We find, using one soft, e.g. CDD, the convex hull of DMUs set, i.e. general convex polyhedron in R^{m+n} given by a system of linear inequalities:

P = { x / A · (x,y) ≤ b }

where A is a p x (m + n) real matrix and b is a real p dimensional vector.

step 3. For every linear inequalities we find the DMUs situated on the (m + n) dimensional plan:

$$P_i : \sum_{j=1}^{m} a_{ij} x_j + \sum_{j=1}^{n} a_{i,m+j} y_j = b_i \quad i = 1,p$$

step 4. We compute for given (x_0,y_0) production the directional distance $D_T(x_0, y_0; g_x, g_y)$ for a certain direction $(g_x,g_y) \geq 0$ direction. If $D_T(x_0, y_0; g_x, g_y) \geq 0$ then (x_0,y_0) is feasible and if $(g_x,g_y) < 0$ then (x_0,y_0) is infeasible.

step 5. If F^i = {DMU / DMU $\in P_i$}, X^i is the card(F^i) × m matrix of inputs of the F^i DMUs and Y^i is the card(F^i) × n matrix of outputs of the F^i DMUs then the program:

$$f_{eff}^{i} = \min_{l} f_{eff}(x_0, y_0; (XY)_i^T l)$$

$$\begin{cases} \sum_{j=1}^{card(F_i)} l_j = 1 \\ X_i^T l \leq x_0 \\ Y_i^T l \geq y_0 \end{cases}$$

$$l \geq 0$$

give the minimum effort necessary for reach the P_i face on a $(g_x,g_y) \geq 0$ direction for a feasible DMU and the program:

$$f_{eff}^{i} = \min_{l} f_{eff}((XY)_i^T l; x_0, y_0)$$

$$\begin{cases} \sum_{j=1}^{card(F_i)} l_j = 1 \\ X_i^T l \geq x_0 \\ Y_i^T l \leq y_0 \end{cases}$$

$$l \geq 0$$

give the minimum effort necessary for obtain (x_0,y_0) technology starting from P_i face on a $(g_x,g_y) \geq 0$ direction, if (x_0,y_0) is infeasible.

step 6. $D_R(x_0,y_0) = f_{eff}^{i_0} = \begin{cases} \min_{i=1,p} f_{eff}^{i} & \text{if } (x_0 y_0) \text{ is feasible} \\ -\min_{i=1,p} f_{eff}^{i} & \text{if } (x_0 y_0) \text{ is infeasible} \end{cases}$

step 7. Projection point of (x_0,y_0) on T frontier is $(x_0^{pp}, y_0^{pp}) = (XY)_{i_0}^T ?_{i_0}^0$ where $?_{i_0}^0$ is the solution of i_0 program.

step 8. Direction of projection is $(g_x^0, g_y^0) = \begin{cases} (x_0 - x_0^{pp}, y_0^{pp} - y_0) & \text{if } (x_0, y_0) \text{ is feasible} \\ (x_0^{pp} - x_0, y_0 - y_0^{pp}) & \text{if } (x_0, y_0) \text{ is infeasible} \end{cases}$

3. Final considerations over the results

The preceding algorithm finds the direction towards the efficient technology easiest has derived but it nothing say who is the route for this.

For an observed (x_0,y_0) DMU a possible information is done by:

$$f_{eff}(x_0, y_0; x_0 + \theta \frac{g_x}{\|g_x, g_y\|}, y_0 + \theta \frac{g_y}{\|g_x, g_y\|})$$

who represent the minimum necessary effort for the reorganisation with θ units par (g_x,g_y) direction starting of (x_0,y_0).

Then $\min_{(g_x g_y) \geq 0} f_{eff}(x_0, y_0; x_0 + \theta \frac{g_x}{\|g_x, g_y\|}, y_0 + \theta \frac{g_y}{\|g_x, g_y\|})$ represent minimum effort for a reorganisation with θ units starting of

(x_0,y_0) and $(g_x^?, g_y^?)$ the corresponding direction.

The value:

$$f_{eff}^q = f_{eff}(x_0, y_0; (x_0 y_0) + D_T(x_0, y_0; g_x^q, g_y^q)\cdot(-g_x^q, g_y^q))$$

represent necessary effort for to realise the efficient technology corresponding of $(g_x^?, g_y^?)$ direction. It is evident that $D_R(x_0,y_0) \le f_{eff}^q$.

The value: $Eff_{AL}^q = \dfrac{f_{eff}^q}{D_R(x_0 y_0)}$

reveal how many times is more difficult to realise the efficient technology corresponding of $(g_x^?, g_y^?)$ direction than easiest variant. Evidently $Eff_{AL}^q \ge 1$.

By and large, for a fixed direction (g_x,g_y) the value:

$$f_{eff(x_0,y_0)}(g_x,g_y) = f_{eff}(x_0, y_0; (x_0 y_0) + D_T(x_0, y_0; g_x, g_y)\cdot(-g_y, g_y))$$

represent the necessary effort to realise the efficient technology corresponding of (g_x,g_y) direction.

If $\{DMU_i \ / \ i = 1,k\}$ is the observed DMU set then: $f_{eff}^T(g_x g_y) = \sum_{i=1}^{k} f_{eff(x_i y_i)}(g_x g_y)$

is the necessary effort that every DMU realise the efficient technology corresponding of (g_x,g_y) direction.

The value: $f_{effopt}^T = \min_{g_x g_y} f_{eff}^T(g_x g_y)$

represent the minimal effort with which every observed DMU realise the efficient technology corresponding of a fixed (g_x,g_y) direction and $(g_x^{T_{min}} g_y^{T_{min}})$ is the direction corresponding of f_{effopt}^T.

If f_{eff}^i is the minimal effort of DMU_i for become efficient and $f_{effopt}^T \ne 0$ then the ratio: $R = \dfrac{\sum_{i=1}^{k} f_{eff}^i}{f_{effopt}^T}$ represent how many is smaller total effort of DMUs to become efficient if each to pertain of itself optimal direction to the variant when the direction is same for everything. Evidently $0 \le R \le 1$.

Because for each DMU corresponds a self-direction is naturally to try to group the observed DMUs to their direction.

A possible variant is to angle of them. If $g^i = (g_x^i, g_y^i)$ is optimal direction of DMU_i and $g^j = (g_x^j, g_y^j)$ for DMU_j then:

$$\frac{\langle g^i g^j \rangle}{\|g^i\| \cdot \|g^j\|} = \cos\alpha$$

where α_{ij} is the measure of angle among the directions of two DMU and $\langle x, y\rangle$, $\|x\|$ and $\|y\|$ are usual notations for the scalar product and the norm of vectors.

Therefore: $\alpha_{ij} = \arccos \dfrac{\langle g^i g^j \rangle}{\|g^i\| \cdot \|g^j\|}$.

and two DMUs are more approaching if α_{ij} is smaller. Then: $F_0^a = \{DMU_i \ / \ \alpha_{0i} \le \alpha\}$ is the set of DMUs of whom optimal directions make an angle less or equal with α.

Also, it is possible to group the DMUs to them effort for become efficient.

However, the most difficult problem remains the estimate of function f_{eff}.

A first observation is than, for the preceding compute, is sufficient to define the function $f_{eff}(x_0,y_0;x,y)$ only for $(x_0 - x, y - y_0) \ge 0$ and, for compute the necessary effort for one fixed DMU (x_0,y_0), is sufficient to define the function $f_{eff}(x_0,y_0;x,y)$ only for the variation of x and y. Also, for compute the minimal necessary effort is sufficient to know the values of $f_{eff}(x_0,y_0;x,y)$ only for (x,y) belonging of convex hull of technology.

4. Examples

Simplest variant is to estimate for each input the necessary effort for reduction with one unit and for each outputs the necessary effort for increase with one unit, to considerate ca the effort is linear in each input and output and that the necessary effort for a combination of reduction and increasing is equal with the sum of the individual efforts.

If (c_x,c_y) represent the vector of the unit effort then the function f_{eff} is:

$$f_{eff}(x_0,y_0;x,y) = (c_x,c_y)^T(x_0 - x, y - y_0)$$

where $(x_0 - x, y - y_0) \geq 0$ and $(c_x,c_y) \geq 0$.

In this case the problem at step 5. is:

$$f^i_{eff} = \min_l (c_x c_y)(x_0 - X_i^T l, Y_i^T l - y_0)$$

$$\begin{cases} \sum_{j=1}^{card(F_i)} l_j = 1 \\ X_i^T l \leq x_0 \\ Y_i^T l \geq y_0 \\ l \geq 0 \end{cases}$$

and it is a linear program.

Moreover, if u = $x_0 - X_i^T l$ and v = $Y_i^T l - y_0$ the problem become:

$$f^i_{eff} = \min_{l,u,v} (c_x^T u + c_y^T v)$$

$$\begin{cases} \sum_{j=1}^{card(F_i)} l_j = 1 \\ X_i^T l + u = x_0 \\ Y_i^T l - v = y_0 \\ l, u, v \geq 0 \end{cases}$$

Because the objective function has the every coefficients positive, the minimum is obtaining when it is a maximal number of u and v equals with 0. Also, because in general the matrix of system have m + n + 1 lines and 2m + 2n colons this number is in general equal of m + n – 1. We would a single u_i or v_j different of 0 so direction of projection is:

$$(g_x^0, g_y^0) = (x_0 - x_0^{pp}, y_0^{pp} - y_0) = (x_0, y_0) - I^p_{m+n}(u_i, v_j)$$

where I^p_{m+n} is a 1 × (m + n) with every components equal of 0 except for the component corresponding of variable u_i or v_j different of 0. Therefore, for a such function of effort, the problem is reducing at the scaling down only one input or the scaling up only one outputs and fixed all other inputs and outputs.

The solution corresponding of direction for which the value of effort functions is minimal in projection point.

An other simple variant is to considerate than effort function depend only at the distance among (x_0,y_0) and (x,y):

$$f_{eff}(x_0,y_0;x,y) = f(\|(x,y) - (x_0,y_0)\|)$$

If this function is linear $f(\|(x,y) - (x_0,y_0)\|) = c \cdot \|(x,y) - (x_0,y_0)\|$, $c \in R_+$ then the problem is reducing at compute geometric distance from (x_0,y_0) at convex hull of technology. Also, she can be increasing or decreasing as the marginal effort is elder or less of 1.

Combining the both variant, we obtain an effort function of type:

$$f_{eff}(x_0,y_0;x,y) = f(\|(c_x,c_y)^T(x_0 - x, y - y_0)\|)$$

However, the problem of estimate of function f_{eff} is open and most probable it depends from peculiarity of each analysed situation.

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